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SUPPRESSION OF MULTIPLE REFLECTIONS.(U)
1978 J LEE, J M MENDEL AFOSR-75-2797

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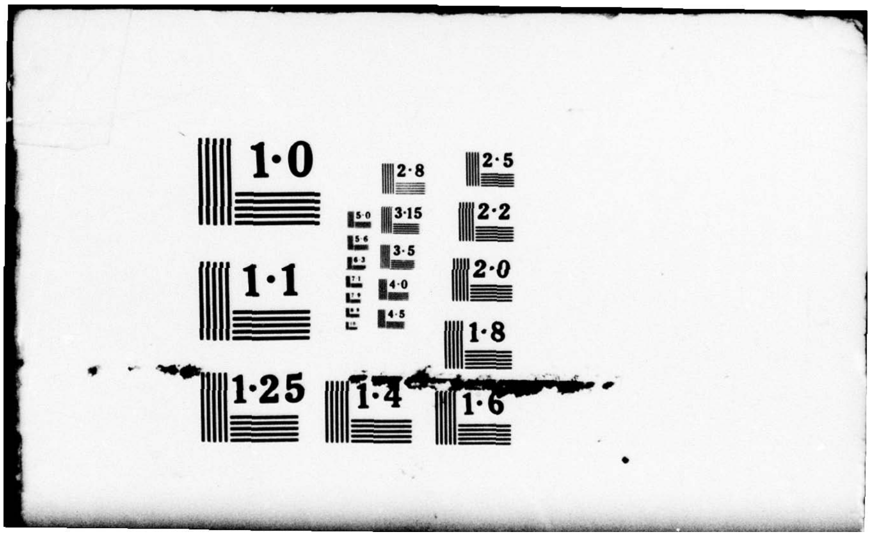
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REPORT DOCUMENTATION PAGE (18)	READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER: AFOSR-TR- 79-0078 — 2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle): (6) SUPRESSION OF MULTIPLE REFLECTIONS.	5. TYPE OF REPORT & PERIOD COVERED: (9) Interim rept.
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): (10) Ja-Sung Lee and Jerry M. Mendel	8. CONTRACT OR GRANT NUMBER(s): (15) AFOSR-75-2797
9. PERFORMING ORGANIZATION NAME AND ADDRESS: University of Southern California, Department of Electrical Engineering, Los Angeles, CA 90007	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 61102F 2304 A1 (16) (17)
11. CONTROLLING OFFICE NAME AND ADDRESS: Air Force Office of Scientific Research/NM, Bolling AFB, Washington, DC 20332	12. REPORT DATE: (11) 1978
	13. NUMBER OF PAGES: 53 (12) 55p.
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report): UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19 KEY WORDS (Continue on reverse side if necessary and identify by block number)

20 ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this paper a simple inverse filter is developed to supress multiple reflections from a normal incidence systhetic seismogram. The filter is developed by means of Mendel's Bremmer Series Decomposition (Refs. 1 & 6)- the state space model for the complete response is decomposed into primaries, secondaries, tertiaries,..., etc., models which generate only primary, secondary, tertiary,..., etc. multiple reflections, respectively. The equation formulations are based on the operator description of state space model of the layered media (ref.2).

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20. Abstract continued.

This filter removes all the multiple reflections from the seismogram which are ever reflected off of the surface, inside the layered media, and is especially effective when the surface reflection coefficient is relatively large as in most geophysical situations. A recursive scheme of applying this filter to the consecutive sybsystems of the layered media is also developed. It generates successive approximations of the primaries of the system, the final result being the pure primaries.

AFOSR-TR- 79-0078

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Box 3098
Tulsa, Oklahama 74101

Preprint subject to later revision, for information of delegates to the SEG Annual Meeting only.

SUPPRESSION OF MULTIPLE REFLECTIONS

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Discussion of this paper is invited immediately after its presentation. Written discussions will be considered for publication in GEOPHYSICS, should the paper be accepted for publication.

1978

ABSTRACT

In this paper a simple inverse filter is developed to suppress multiple reflections from a normal incidence synthetic seismogram. The filter is developed by means of Mendel's Bremmer Series Decomposition (Refs. 1 and 7) - the state space model for the complete response is decomposed into primaries, secondaries, tertiaries,..., etc., models which generate only primary, secondary, tertiary, ..., etc. multiple reflections, respectively. The equation formulations are based on the operator description of state space model of the layered media (ref. 2).

This filter removes all the multiple reflections from the seismogram which are ever reflected off of the surface, inside the layered media, and is especially effective when the surface reflection coefficient is relatively large as in most geophysical situations. A recursive scheme of applying this filter to the consecutive subsystems of the layered media is also developed. It generates successive approximations of the primaries of the system, the final result being the pure primaries.

I. Introduction

In seismic profiling of layered media, primary reflections - compressional waves reflected directly from the interfaces of the layers - contain the ultimate information for the determination of the subsurface structure of the media. Minimization of the contribution from other undesirable events, such as noise, surface waves and ghost reflections, to maximize the contribution from primary reflection has been one of the fundamental problems in the analysis of seismic traces. Another and particularly troublesome type of interference is that from multiple reflections. There are some special classes of multiple reflections which exhibit significant amplitudes so that they occur well into the seismogram and affect its appearance. They are most likely to happen when the layered system involves one or more strong reflectors. They occur through reinforcement of several multiple reflections. A general theory has been developed to locate and identify various kinds of multiple reflections and reinforced events between them (ref. 6). In the analysis of multiple reflections, which represent a highly complicated mechanism of layered media, Mendel's Bremmer Series Decomposition (refs. 1 and 7) has been found to be useful, which makes it possible to study each of the decomposed multiple reflections such as primaries, secondaries, tertiaries, ..., etc., separately. The state space models for lossless waves in layered media which are described by the wave equations and boundary conditions have been developed by Mendel, et. al. (refs. 2 and 3). The models are for non-equal one-way travel times and, by the nature of non-uniformity, represent a special class of equations with multiple delays which are referred to as causal functional equations. Based on these equations Mendel (ref. 1) has proven the truth of the following decomposition of the solution to the lossless wave equation in layered media : the complete output from a K-layer media system, which is comprised of the superposition of primaries, secondaries, tertiaries, etc., can be obtained from a single state space model of order 2K - the complete model - or from an infinite number of models, each of order 2K, the output of the first of which is just the primaries, the output of the second of which is just the secondaries, etc. This decomposition of the solution to the lossless wave equation into physically meaningful constituents (i.e., primaries, secondaries, etc.) is called a Bremmer Series Decomposition, after Bremmer, who in 1951 (ref. 4) established a similar decomposition.

In this paper, by means of Bremmer Series Decomposition, we develop some methods to approximate the primary reflections of a layered system from the normal incident synthetic seismogram generated by the system for the given information, such as surface reflection coefficient and input waveform, etc. From these a simple inverse filter is derived which suppresses multiple reflections. The filter removes all the multiple reflections from the seismogram which are ever reflected off of the surface inside the media. This filter is especially effective when the surface reflection coefficient is relatively large as in most geophysical situations. A recursive method is also developed which applies the filter to the consecutive subsystems of the layered media from the top to the bottom layer, generating successive approximations of the primaries of the system.

In the first few sections in this paper (from section II through section V), we review the state space model, the Bremmer Series Decomposition and its operational and transfer functional descriptions. The inverse filter, which is denoted by F_1 throughout this paper, is developed in Section VI. In Section VII, another inverse filter is introduced which is derived from a further approximation of primaries with some additional information of the layered system. The effects of those filters and their relationship are examined in Section VIII, from which the recursive method of applying filter F_1 to the subsystems of the media is developed in Section IX. For illustrative purpose, a three layer example is used throughout the analysis.

II. A State Space Model

A state space model for the system of K-layered media, depicted in Figure 1, is derived under the following modeling assumptions; (1) plane compressional waves, (2) horizontally stratified nonabsorptive layers of different travel times, and (3) normally incident waves. Each layer is characterized by its one way travel time, τ_i, velocity V_i, and normal incidence reflection coefficient r_i $(i = 1,2,\ldots,K)$. Additionally, interface-0 denotes the surface and is characterized by reflection coefficient r_0. We adopt the convention of calling the layer below layer K the basement. In Figure 1, m(t) and y(t) denote the input (e.g., seismic source signature from dynamite, airgun, etc.) to the layered earth system which is applied at interface-0, and the output (i.e., ideal seismogram) of the system which is observed at the surface respectively.

The compressional waves within the k-th layer are identified by two states u_k and d_k, which denote the upgoing and downgoing waves in the k-th layers, respectively. At present time t, u_k is defined at the top of layer k, whereas d_k is defined at the bottom of layer k, as shown in Figure 2. To develop the state equation model we direct our attention at the intersection point of the ray diagram and apply superposition to obtain the following equations for signals $u_k(t+\tau_k)$ and $d_{k+1}(t+\tau_{k+1})$, which leave that point;

$$u_k(t+\tau_k) = r_k\, d_k(t) + (1-r_k)\, u_{k+1}(t) \tag{1a}$$

$$d_{k+1}(t+\tau_{k+1}) = (1+r_k)\, d_k(t) - r_k\, u_{k+1}(t) \tag{1b}$$

These equations are applicable for k=1,2,...,K-1. At the surface (Figure 3a), we obtain

$$d_1(t+\tau_1) = -r_0\, u_1(t) + (1+r_0)\, m(t) \tag{2}$$

$$y(t) = (1-r_0)\, u_1(t) \tag{3}$$

and*, at the K-th interface, we assume that $u_{K+1}(t)=0$, to obtain (Figure 3b)

$$u_K(t+\tau_K) = r_K\, d_K(t) \tag{4}$$

Signal $y(t)$ in Eq. (3) is the measurable system output. Signal $d_{K+1}(t)$, which is the downgoing wave in the basement, is also a system output; but, since it cannot be measured, we shall ignore it in the following analyses.

It is convenient to group Eqs. (1a), (1b), (2) and (4) in a <u>layer ordering</u>, as follows :

$$\begin{aligned}
d_1(t+\tau_1) &= -r_0\, u_1(t) + (1+r_0)\, m(t)\\
u_1(t+\tau_1) &= r_1\, d_1(t) + (1-r_1)\, u_2(t)\\
&\left.\begin{aligned}
d_j(t+\tau_j) &= (1+r_{j-1})\, d_{j-1}(t) - r_{j-1}\, u_j(t)\\
u_j(t+\tau_j) &= r_j\, d_j(t) + (1-r_j)\, u_{j+1}(t)
\end{aligned}\right\}\ j=2,3,\ldots,K-1\\
d_K(t+\tau_K) &= (1+r_{K-1})\, d_{K-1}(t) - r_{K-1}\, u_K(t)\\
u_K(t+\tau_K) &= r_K\, d_K(t)
\end{aligned} \tag{5}$$

Equations (5) and (3) together represent the state equation model for the complete output $y(t)$; hence, they are referred to in the sequel as the <u>complete model</u>.

<u>Comment</u>. Equations (5) and (3) can be expressed in more compact notation by introducing the following $2K \times 2K$ matrix operators :

$$\mathcal{Z} \triangleq \mathrm{diag}\,(z_1, z_1, z_2, z_2, \ldots, z_K, z_K), \tag{6}$$

where z_i is a scalar operator used to denote a τ_i sec. time delay (i.e., $z_i(t) = f(t-\tau_i)$). Let

$$\mathcal{X}(t) = \mathrm{col}\,(u_1(t), d_1(t), u_2(t), d_2(t), \ldots, u_K(t), d_K(t)) ;$$

then Eqs. (5) and (3) can be written, as

$$\mathcal{Z}^{-1}\mathcal{X}(t) = A\,\mathcal{X}(t) + \underline{b}\, m(t) \tag{7a}$$

$$y(t) = \underline{c}'\mathcal{X}(t) \tag{7b}$$

---

* In refs. 1,2,3,6 and 7, the output equation was defined as $y(t) = (1-r_0)\, u_1(t) + r_0\, m(t)$ including the direct reflection term $r_0\, m(t)$. Here we neglect the direct reflection term.

where the explicit structures of A, $\underline{b}$ and $\underline{c}$ can be deduced directly from the former equations. State Eq. (7a) is a dynamical equation with multiple time delays. It is not a differential equation, nor is it a finite-difference equation. We shall refer to it as a <u>causal functional equation</u>. It is linear and time-invariant, and, as is the case with delay-time systems, requires initial value information over initial intervals of time. Consider the k-th layer (Figure 2). Then $d_k(t)$ is equal to zero until $t = \tau_1 + \tau_2 + \ldots + \tau_k$, and u_k is equal to zero until $t = \tau_1 + \tau_2 + \ldots + 2\tau_k$. These facts are true for all $k = 1,2,\ldots,K$; i.e.,

$$d_j(t) = 0 \qquad \forall \quad t \in \Big[0, \sum_{i=1}^{j} \tau_i\Big) \tag{8a}$$

and

$$u_j(t) = 0 \qquad \forall \quad t \in \Big[0, \sum_{i=1}^{j} \tau_i + \tau_j\Big) \tag{8b}$$

where $j = 1,2,\ldots,K$.

For more detailed discussions about the derivation and structure of the state space model, the reader is referred to refs. 2 and 3.

III. A Bremmer Series Decomposition

Since the complete output $y(t)$ is a superposition of primaries, secondaries, tertiaries, etc., it can be written as

$$y(t) = \sum_{j=1}^{\infty} y_j(t) \ , \tag{9}$$

where $y_j(t)$ denotes the j-ary reflection components of $y(t)$. In this paper we just summarize this decomposition in the following theorem, given without proof (see refs. 1 and 3 for the proof).

Theorem

The complete output, $y(t)$, from a K-layer media system can be obtained from a single model of order 2K - the complete model, given by Eqs. (5) and (3) - or from an infinite number of models, each of order 2K, interconnected as shown in Figure 4. The primaries model, which generates only primary reflections, and the n-aries models, each of which generates only n-ary reflections, where $n = 2,3,\ldots$, are defined by the following :

(a) Primaries model

$$\begin{aligned}
d_{1,1}(t+\tau_1) &= (1+r_0)\, m(t) \\
u_{1,1}(t+\tau_1) &= r_1\, d_{1,1}(t) + (1-r_1)\, u_{1,2}(t) \\
\left.\begin{aligned} d_{1,j}(t+\tau_j) &= (1+r_{j-1})\, d_{1,j-1}(t) \\ u_{1,j}(t+\tau_j) &= r_j\, d_{1,j}(t) + (1-r_j)\, u_{1,j+1}(t) \end{aligned}\right\} & \ j = 2,3,\ldots,K-1 \\
d_{1,K}(t+\tau_K) &= (1+r_{K-1})\, d_{1,K-1}(t) \\
u_{1,K}(t+\tau_K) &= r_K\, d_{1,K}(t)
\end{aligned} \tag{10}$$

and

$$y_1(t) = (1-r_0)\, u_{1,1}(t) \tag{11}$$

(b) n-aries model

$$d_{n,1}(t+\tau_1) = -r_0\, u_{n-1,1}(t)$$

$$u_{n,1}(t+\tau_1) = r_1\, d_{n,1}(t) + (1-r_1)\, u_{n,2}(t)$$

$$\left.\begin{aligned} d_{n,j}(t+\tau_j) &= (1+r_{j-1})\, d_{n,j-1}(t) - r_{j-1}\, u_{n-1,j}(t) \\ u_{n,j}(t+\tau_j) &= r_j\, d_{n,j}(t) + (1-r_j)\, u_{n,j+1}(t) \end{aligned}\right\} j=2,3,\ldots,K-1$$

$$d_{n,K}(t+\tau_K) = (1+r_{K-1})\, d_{n,K-1}(t) - r_{K-1}\, u_{n-1,K}(t)$$

$$u_{n,K}(t+\tau_K) = r_K\, d_{n,K}(t) \tag{12}$$

and

$$y_n(t) = (1-r_0)\, u_{n,1}(t) \tag{13}$$

for $n=2,3,\ldots$ Additionally, $y(t)$ is given by Eq. (9). In these equations, $d_{n,j}$, $u_{n,j}$ and y_n denote n-ary downgoing and upgoing states and the n-ary reflection portion of the complete output, respectively.

The primaries model is obtained from the complete model given by Eq. (5) by deleting the term $-r_{j-1}\, u_j(t)$ in the equations for $d_j(t+\tau_j)$, $j=1,2,\ldots,K$. This is done to truncate the multiple reflections higher than secondary reflections which are due to the upgoing waves reflecting off of the top of the layers. The n-aries model is obtained in a similar manner, by successively subtracting all the j-aries models (where $j=1,2,\ldots,n-1$), from the complete model to obtain a residual model, and, by then deleting the terms which truncate multiple reflections higher than n-ary reflections in that residual model. The reader who is interested in the detailed derivation of the n-aries model is referred to ref. 1 or 7.

Instead of a formal proof of the above theorem, we demonstrate its validation through a three-layer simulation (Figures 5 and 6). Figure 5 depicts a three layer media which can be associated with a bright spot phenomena, because of the thin low velocity layer which is sandwiched in between the two thick high velocity layers. We attribute no other geological plausibility to this example. For layer 1, $V_1 = 7{,}500$ ft/sec and $\rho_1 = 2{,}2$ gm/cm^3; for layer 2, $V_2 = 5{,}500$ ft/sec and $\rho_2 = 1{,}6$ gm/cm^3; for layer 3, $V_3 = V_1$ and $\rho_3 = \rho_1$; for the basement, $V_4 = 12{,}000$ ft/sec and ρ_4 was approximated by the 1/4 - power law, $0{,}23\, V_4^{1/4}$ (ref. 5).

Figure 6 depicts the complete response as well as the primaries, secondaries and some of the tertiaries (through 2 seconds) which were obtained via simulation of Eqs. (10)-(13).

In this example we observe that the superposition of the first three terms in the Bremmer series decomposition is a good approximation to the complete response. In many geophysical situations, where reflection coefficients are quite small, the decomposition can be truncated after secondaries or tertiaries; hence, the Bremmer series decomposition also represents a way to approximate the solution to the wave equation. This, together with the fact that, by means of the decomposition, it is possible to deal with each of the multiple reflections separately, simplifies complicated problems in the analyses of a seismogram.

IV. Operator Descriptions for Bremmer Series Decomposition and Layer Transition Matrix H

The state space equations (10) for the primaries model can be expressed in a compact way by introducing the following $K \times K$ matrix operator,

$$Z \stackrel{\Delta}{=} \operatorname{diag}\,(z_1, z_2, \ldots, z_K), \tag{14}$$

where z_i is a scalar operator used to denote a τ_i sec. time delay (i.e. $z_i\, f(t) = f(t-\tau_i)$), and, by reordering the equations in such a manner that all downgoing states are grouped together and all upgoing states are grouped together. Let

$$\underline{d}_1(t) = \operatorname{col}\,(d_{1,1}(t),\ d_{1,2}(t),\ \ldots,\ d_{1,K}(t)) \tag{15a}$$

$$\underline{u}_1(t) = \operatorname{col}\,(u_{1,1}(t),\ u_{1,2}(t),\ \ldots,\ u_{1,K}(t)). \tag{15b}$$

Then, Equations (10) can be written, in partitioned form as :

$$Z^{-1}\,\underline{d}_1(t) = A_1\,\underline{d}_1(t) + \underline{g}\,m(t) \tag{16a}$$

$$Z^{-1}\,\underline{u}_1(t) = A_3\,\underline{d}_1(t) + A_4\,\underline{u}_1(t)$$

where

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ (1+r_1) & 0 & 0 & \cdots & 0 & 0 \\ 0 & (1+r_2) & 0 & \cdots & 0 & 0 \\ 0 & 0 & (1+r_3) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & (1+r_{K-1}) & 0 \end{bmatrix} \tag{17}$$

$$A_3 = \operatorname{diag}\,(r_1, r_2, \ldots, r_K) \tag{18}$$

$$A_4 = \begin{bmatrix} 0 & (1-r_1) & 0 & 0 & \cdots & 0 \\ 0 & 0 & (1-r_2) & 0 & \cdots & 0 \\ 0 & 0 & 0 & (1-r_3) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (1-r_{K-1}) \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{19}$$

and

$$\underline{g} = \text{col}\,(1 + r_0, 0, 0, \ldots, 0) \ . \tag{20}$$

Matrices A_1, A_3 and A_4 are $K \times K$, and $\underline{g}$ is $K \times 1$. Equations (16) can be solved for $\underline{u}_1(t)$ as

$$\underline{u}_1(t) = (I - ZA_4)^{-1}\ ZA_3(I - ZA_1)^{-1}\ Z\underline{g}\ m(t) \ . \tag{21}$$

In addition to Eq. (21), we have the observation Eq. (11), which can be written as

$$y_1(t) = (0,0,\ldots,0 \mid (1 - r_0),0,\ldots,0) \begin{pmatrix} \underline{d}_1(t) \\ \underline{u}_1(t) \end{pmatrix}$$

or

$$y_1(t) = \underline{h}'\ \underline{u}_1(t) \tag{22}$$

where

$$\underline{h}' = ((1 - r_0),0,\ldots,0), \text{ which is } 1 \times K \ . \tag{23}$$

In the same manner, Eq. (12) for n-aries model can be expressed as

$$Z^{-1}\ \underline{d}_1(t) = A_1\ \underline{d}_n(t) + A_2\ \underline{u}_{n-1}(t) \tag{24a}$$

$$Z^{-1}\ \underline{u}_n(t) = A_3\ \underline{d}_n(t) + A_4\ \underline{u}_n(t) \tag{24b}$$

where A_2 is a $K \times K$ matrix given by

$$A_2 = \text{diag}\,(-r_0,-r_1,\ldots,-r_{K-1}) \tag{25}$$

We notice that in Eq. (24a), the input to the system of n-aries model is the upgoing waves, $\underline{u}_{n-1}(t)$ of the (n-1)-aries model. Solving Eqs. (24a) and (24b) for $\underline{u}_n(t)$, we find that

$$\underline{u}_n(t) = (I - ZA_4)^{-1}\ ZA_3(I - ZA_1)^{-1}\ ZA_2\ \underline{u}_{n-1}(t) \ . \tag{26}$$

Additionally (13) can be expressed as

$$y_n(t) = \underline{h}'\ \underline{u}_n(t) \tag{27}$$

where $\underline{h}'$ is given by (23)

We observe that the same matrix term $(I - ZA_4)^{-1} ZA_3(I - ZA_1)^{-1}Z$ appears in Eqs. (21) and (26). Let

$$H \triangleq (I - ZA_4)^{-1} ZA_3(I - ZA_1)^{-1} Z . \tag{28}$$

Then, Eqs. (21) and (26) can be expressed in terms of H, as

$$\underline{u}_1(t) = H \, \underline{g} \, m(t) \tag{29a}$$

$$\underline{u}_n(t) = H \, A_2 \, \underline{u}_{n-1}(t) \tag{29b}$$

These are the recursive equations for $\underline{u}_n(t)$. In non-recursive form, we can write Eq. (29b) as

$$\underline{u}_n(t) = (HA_2)^{n-1} \, \underline{u}_1(t) . \tag{30}$$

Explicit expressions for matrix H in terms of reflection and transmission coefficients with multiple delay operators were obtained in ref. 6. Matrix H is $K \times K$, for a K-layer system, and is given by

$$H = \{h_{ij}\}$$

$$h_{ij} = \left[\sum_{n=i}^{K} r_n \, s_{n-1} \, e_n\right] \Big/ \sqrt{e_{i-1} \, e_{j-1}} \, p_{i-1} \, q_{j-1} \tag{31a}$$

for $i \geq j$, $i,j = 1,2,\ldots,K$

and

$$h_{ij} = h_{ji}\Big|_{p \leftrightarrow q} \tag{31b}$$

where

$$e_i = \prod_{\ell=1}^{i} z_\ell^2 , \quad s_i = \prod_{\ell=1}^{i} (1 - r_\ell^2) , \quad p_i = \prod_{\ell=1}^{i} (1 - r_\ell)$$

$$q_i = \prod_{\ell=1}^{i} (1 + r_\ell), \text{ and } s_0 = p_0 = q_0 = 1.$$

For example, when $K = 3$, H is given by

$$H = \begin{bmatrix} z_1^2 r_1 + z_1^2 z_2^2 r_2 (1 - r_1^2) + z_1^2 z_2^2 z_3^2 r_3 (1 - r_1^2)(1 - r_2^2) & z_1 z_2^2 r_2 (1 - r_1) + z_1 z_2^2 z_3^2 r_3 (1 - r_1)(1 - r_2^2) & z_1 z_2 z_3^2 r_3 (1 - r_1)(1 - r_2) \\ z_1 z_2^2 r_2 (1 + r_1) + z_1 z_2^2 z_3^2 r_3 (1 + r_1)(1 - r_2^2) & z_2^2 r_2 + z_2^2 z_3^2 r_3 (1 - r_2^2) & z_2 z_3^2 r_3 (1 - r_2) \\ z_1 z_2 z_3^2 r_3 (1 + r_1)(1 + r_2) & z_2 z_3^2 r_3 (1 + r_2) & z_3^2 r_3 \end{bmatrix} \triangleq \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \tag{32}$$

Physically, element h_{ij} of matrix H represents the unit impulse response of single bounce reflections observed at the top of the i-th layer due to a unit impulse excited downward from the top of the j-th layer. Figure 7 illustrates some examples. We refer to matrix H as the <u>layer transition matrix</u>.

V. Transfer Function Representations of Bremmer Series Decomposition

Taking the Laplace Transformation of Eqs. (10), (11), (12) and (13), and introducing the following $K \times K$ matrix,

$$Z(s) \triangleq \text{diag}\left[e^{-\tau_1 s}, e^{-\tau_2 s}, \ldots, e^{-\tau_K s}\right] \tag{33}$$

we obtain equations of the same structure as Eqs. (16a), (16b), (24a) and (24b); i.e.,

$$Z^{-1}(s)\,\underline{D}_1(s) = A_1\,\underline{D}_1(s) + \underline{g}\,M(s) \tag{34a}$$

$$Z^{-1}(s)\,\underline{U}_1(s) = A_3\,\underline{D}_1(s) + A_4\,\underline{U}_1(s) \tag{34b}$$

for primaries model, and

$$Z^{-1}(s)\,\underline{D}_n(s) = A_1\,\underline{D}_n(s) + A_2\,\underline{U}_{n-1}(s) \tag{35a}$$

$$Z^{-1}(s)\,\underline{U}_n(s) = A_3\,\underline{D}_n(s) + A_4\,\underline{U}_n(s) \tag{35b}$$

for n-aries model, where

$$\underline{D}_n(s) = \text{col}\,(D_{n,1}(s), D_{n,2}(s), \ldots, D_{n,K}(s)) \tag{36a}$$

$$\underline{U}_n(s) = \text{col}\,(U_{n,1}(s), U_{n,2}(s), \ldots, U_{n,K}(s)) \tag{36b}$$

$n = 1,2,\ldots,K$.

It is straight forward to show that the Laplace transforms of the rest of the equations in Section IV also remain the same in their explicit structures. Especially, the Laplace transforms of Eqs. (29a), (29b) and (30) are given by

$$\underline{U}_1(s) = H(s)\,\underline{g}\,M(s) \tag{37a}$$

$$\underline{U}_n(s) = H(s)\,A_2\,\underline{U}_{n-1}(s) \tag{37b}$$

and

$$\underline{U}_n(s) = (H(s)\,A_2)^{n-1}\,\underline{U}_1(s) \tag{38}$$

respectively, where

$$H(s) = (I - Z(s)\,A_4)^{-1}\,Z(s)\,A_3(I - Z(s)\,A_1)^{-1}\,Z(s) \tag{39}$$

which is the Laplace transformed version of the layer transition matrix H given by Eq. (28).

In addition, the output equation is given by, from Eqs. (22) and (27),

$$Y_n(s) = \underline{h}' \, \underline{U}_n(s) \tag{40}$$

$n = 1,2,\ldots,K.$

Z-transformation is also applicable in our Bremmer series decomposition if we let

$$\tau_i = n_i T \, , \; i = 1,2,\ldots,K \tag{41}$$

where T is the sampling interval. As we did in Laplace transformation, defining

$$Z(z) \triangleq \text{diag}\left[z^{n_1}, z^{n_2}, \ldots, z^{n_K}\right] , \tag{42}$$

we obtain z-transformed version of Eqs. (16a), (16b), (24a) and (24b); i,e.,

$$Z^{-1}(z) \, \underline{D}_1(z) = A_1 \, \underline{D}_1(z) + \underline{g} \, M(z) \tag{43a}$$

$$Z^{-1}(z) \, \underline{U}_1(z) = A_3 \, \underline{D}_1(z) + A_4 \, \underline{U}_1(z) \tag{43b}$$

for primaries model, and

$$Z(z)^{-1} \, \underline{D}_n(z) = A_1 \, \underline{D}_n(z) + A_2 \, \underline{U}_{n-1}(z) \tag{44a}$$

$$Z(z)^{-1} \, \underline{U}_n(z) = A_3 \, \underline{D}_n(z) + A_4 \, \underline{U}_n(z) \tag{44b}$$

for n-aries model, where

$$\underline{D}_n(z) = \text{col}\,(D_{n,1}(z), D_{n,2}(z), \ldots, D_{n,K}(z)) \tag{45a}$$

$$\underline{U}_n(z) = \text{col}\,(U_{n,1}(z), U_{n,2}(z), \ldots, U_{n,K}(z)) \tag{45b}$$

$N = 1,2,\ldots,K.$

The z-transforms of the rest of the equations in Section IV also remain the same in their explicit structures. Consequently, the operational representations in Section IV, the Laplace transforms and the z-transforms are readily transferrable to each other merely by changing the notations of corresponding variables.

The corresponding z-transforms of Eqs. (29a), (29b) and (30) are given by

$$\underline{U}_1(z) = H(z)\,\underline{g}\,M(z) \tag{46a}$$

$$\underline{U}_n(z) = H(z)\,A_2\,\underline{U}_{n-1}(z) \tag{46b}$$

and

$$\underline{U}_n(z) = (H(z)\,A_2)^{n-1}\,\underline{U}_1(z) \tag{47}$$

respectively, where

$$H(z) = (I - Z(z)\,A_4)^{-1}\,Z(z)\,A_3\,(I - Z(z)\,A_1)^{-1}\,Z(z). \tag{48}$$

In addition, the output equations are

$$Y_n(z) = \underline{h}'\,\underline{U}_n(z) \tag{49}$$

$n = 1,2,\ldots,K.$

It is not difficult to show that

$$H(s) = H\Big|_{z_i \to e^{-\tau_i s}} \tag{50a}$$

$$H(z) = H\Big|_{z_i \to z^{n_i}} \tag{50b}$$

$i = 1,2,\ldots,K$. We refer to $H(s)$ and $H(z)$ as the <u>layer transfer matrix</u>.

VI. Derivation of the Inverse Filter for Suppressing Multiple Reflections

In the following analysis we restrict ourselves to the system of layered media for which the modeling assumptions given in Section II apply. We will use the z-transform representations of our state space model in all derivations throughout this analysis; for, it is usually convenient for simulation purposes.

Our objective is : given a synthetic seismogram with some information about the system, (such as estimated reflection coefficients and input waveform) to suppress multiple reflections from the seismogram so as to maximize the contributions from the primaries. In our state space model, it is equivalent to say that given $Y(z)$, the complete response of the layered media, we want to find an inverse filter which gives $Y_1(z)$, the primaries (the output of the primaries model), or an approximation of $Y_1(z)$ from $Y(z)$.

Suppose the reflection coefficient, r_0, of the surface and the input waveform, $M(z)$, are both known, and that the seismogram $Y(z)$ (the output of the system generated by $M(z)$) is also known. We are going to suppress multiple reflections from $Y(z)$ as much as possible making use only of r_0, $M(z)$ and $Y(z)$.

Let

$$\underline{U}(z) = (U_1(z), U_2(z), \ldots, U_K(z)) \tag{51}$$

where $U_i(z)$ is the z-transform of $u_i(t)$ which is the upgoing wave in the i-th layer of the complete model. Since the response of the complete model is the superposition of the responses of primaries, secondaries, tertiaries, etc. models, we can write

$$\underline{U}(z) = \sum_{n=1}^{\infty} \underline{U}_n(z) \tag{52}$$

where $\underline{U}_n(z)$ is given in Eq. (45b). Notice the difference between $U_i(z)$ and $\underline{U}_i(z)$; the former is a scalar function which is associated with the i-th layer in the complete model, whereas the latter is a vector which denotes all the upgoing waves in the i-aries model.

Substituting Eq. (41) into (52), we have

$$\underline{U}(z) = \sum_{n=1}^{\infty} (HA_2)^{n-1} \, \underline{U}_1(z) \tag{53}$$

where the explicit dependence of H on z is omitted for notational simplicity. Since the infinite series in the right hand side of Eq. (53) converges to $\underline{U}(z)$, we can write

$$\sum_{n=1}^{\infty} (HA_2)^{n-1} = \left[I - HA_2\right]^{-1} \tag{54}$$

Hence,

$$\underline{U}(z) = \left[I - HA_2\right]^{-1} \underline{U}_1(z) \tag{55}$$

or

$$\underline{U}_1(z) = \left[I - HA_2\right] \underline{U}(z) \tag{56}$$

This is the equation for the upgoing primary waves in terms of the upgoing waves of the complete model. Let

$$H = \{\hat{h}_{ij}\} \qquad i,j = 1,2,\ldots,K \, . \tag{57a}$$

Then $\hat{h}_{ij}$ is the corresponding z-transform of h_{ij} given by Eq. (31); i.e.,

$$\hat{h}_{ij} = h_{ij}\Big|_{z_i \to z^{n_i}} \tag{57b}$$

Then, substituting (57b) into the vector equation (56) and taking its first component, we have

$$U_{1,1}(z) = (1 + r_0 \, \hat{h}_{11}) \, U_1(z) + r_1 \, \hat{h}_{12} \, U_2(z) + \ldots + r_{K-1} \, \hat{h}_{1K} \, U_k(z)$$

or

$$U_{1,1}(z) = U_1(z) - \left[-r_0 \, \hat{h}_{11} \, U_1(z) - r_1 \, \hat{h}_{12} \, U_2(z) - \ldots - r_{k-1} \, \hat{h}_{1K} \, U_k(z)\right] \tag{58}$$

Since $U_{1,1}(z)$ represents the primary reflections and $U_1(z)$ represents the complete response, the terms in the brackets in Eq. (58) should represent all the multiple reflections. Our objective is to suppress those from $U_1(z)$. To achieve this purpose we should express the bracketed terms in terms of $U_{1,1}(z)$ and our known quantities $U_1(z)$, r_0 and $M(z)$ ($U_1(z)$ is obtained directly from $Y(z)$ by Eq. (3)). But, this is impossible because those terms in the brackets are functions of $\tau_1,\tau_2,\ldots,\tau_k$ and $r_1,r_2,\ldots,r_K$ which we do not know and moreover $U_2(z)$, $U_3(z)$, ..., $U_K(z)$ are not measurable. However, the first term in the brackets, especially $\hat{h}_{11}$, which is also a function of $\tau_1,\tau_2,\ldots,\tau_K$ and $r_1,r_2,\ldots,r_K$, can be expressed in terms of $U_{1,1}(z)$ and the known quantities from Eq. (46a), and this fact enables us to make an approximation of the primaries $U_{1,1}(z)$ from $U_1(z)$.

Taking the first component of the vector equation (46a), we have

$$U_{1,1}(z) = \hat{h}_{11}(1+r_0)\,M(z) \tag{59}$$

Hence,

$$\hat{h}_{11} = \frac{U_{1,1}(z)}{(1+r_0)\,M(z)} \tag{60}$$

Substituting this into Eq. (58),

$$U_{1,1}(z) = U_1(z) - \left[-r_0\,\frac{U_{1,1}(z)}{(1+r_0)\,M(z)}\,U_1(z) + \alpha(z)\right] \tag{61}$$

where

$$\alpha(z) = -r_1\,\hat{h}_{12}\,U_2(z) - \ldots - r_{K-1}\,\hat{h}_{1K}(z) \tag{62}$$

From Eq. (61), it follows that

$$U_{1,1}(z) = \frac{U_1(z)}{1-\dfrac{r_0}{(1+r_0)}\dfrac{U_1(z)}{M(z)}} - \frac{\alpha(z)}{1-\dfrac{r_0}{(1+r_0)}\dfrac{U_1(z)}{M(z)}} \tag{63}$$

From Eqs. (3) and (11), we have

$$U_1(z) = \frac{1}{(1-r_0)}\,Y(z) \tag{64}$$

$$U_{1,1}(z) = \frac{1}{(1-r_0)}\,Y_1(z). \tag{65}$$

Substituting these into Eq. (63) for $U_1(z)$ and $U_{1,1}(z)$, we get

$$Y_1(z) = \frac{Y(z)}{1 - \dfrac{r_0}{(1-r_0^2)}\dfrac{Y(z)}{M(z)}} - \frac{(1-r_0)\,\alpha(z)}{1 - \dfrac{r_0}{(1-r_0^2)}\dfrac{Y(z)}{M(z)}} \tag{66}$$

Usually the second term in the right-hand side of Eqs. (66) is quite small in magnitude compared with $Y_1(z)$, and especially when $r_o >> r_i$ $(i = 1,2,\ldots,K)$, as in most geophysical situations, this term is almost negligible. In this case

$$Y_1(z) \approx \frac{Y(z)}{1 - \dfrac{r_0}{(1-r_0^2)}\dfrac{Y(z)}{M(z)}} \tag{67}$$

Let

$$F_1(z) = \frac{Y(z)}{1 - \dfrac{r_0}{(1-r_0^2)}\dfrac{Y(z)}{M(z)}} \tag{68}$$

and

$$\beta(z) = \frac{(1-r_0)\,\alpha(z)}{1 - \dfrac{r_0}{(1-r_0^2)}\dfrac{Y(z)}{M(z)}} \tag{69}$$

Then Eq. (66) becomes

$$Y_1(z) = F_1(z) - \beta(z)$$

or

$$F_1(z) = Y_1(z) + \beta(z) \tag{70}$$

Let the relation given by Eq. (68) be F_1. Then F_1 is the inverse filter we were looking for; i.e., given the synthetic seismogram $Y(z)$ with knowledge of the surface reflection coefficient, r_0, and the input waveform, $M(z)$, F_1 suppresses some amount of (actually the most significant portions of) the multiple reflections from $Y(z)$ and yields an approximation of the primaries $Y_1(z)$.

If the input is an impulse, then $M(z) = 1$ and Eq. (68) reduces to

$$F_1(z) = \frac{Y(z)}{1 - \frac{r_0}{(1-r_0^2)} Y(z)} \tag{71}$$

In this case $Y(z)$ represents the transfer function of the layered media.

A simulated result is shown in Figures 8 through 10 for the three layer example given in Section III. The impulse response of the media and its filtered output, obtained by Eq. (71), are shown together in Figure 8. Figure 9 depicts the input waveform and Figure 10(a) is the synthetic seismogram due to this input convolved with the impulse response. The result of applying the filter in Eq. (68) to the seismogram is shown in Figure 10(b). The three prominent peaks in Figure 10(b) represent the primaries $Y_1(z)$. The remaining small ripples represent the additional term $\beta(z)$ in Eq. (69) or (70). We see that, in this example, significant multiple reflections are almost completely eliminated by our filter.

VII. Another Filter, which makes use of Additional Information about r_1 and τ_1.

Suppose the reflection coefficient r_1 associated with the 1st interface of the media and the one way travel time τ_1 of the 1st layer are known in addition to knowledge of r_0 and the input waveform. This assumption may be applicable in the marine situation. Making use of this extra information we want to find an inverse filter which suppresses some additional multiple reflections.

In the following we will show that the second term $-r_1 \hat{h}_{12} U_2(z)$ in the brackets in Eq. (58) can be expressed in terms of $U_{1,1}(z)$ and the known quantities r_0, r_1, τ_1 and $M(z)$.

From Eqs. (31a), (31b) and (57b), we have

$$\frac{\hat{h}_{ji}}{\hat{h}_{ij}} = \frac{p_{i-j}}{q_{i-j}} \quad \text{for } i \geq j \tag{72}$$

Especially,

$$\frac{\hat{h}_{12}}{\hat{h}_{21}} = \frac{p_1}{q_1} = \frac{(1-r_1)}{(1+r_1)} \tag{73}$$

or

$$\hat{h}_{12} = \frac{(1-r_1)}{(1+r_1)} \hat{h}_{21} \tag{74}$$

Now, taking the second component of the vector equation (46a), we have

$$\hat{h}_{21} = \frac{U_{1,2}(z)}{(1+r_0)\, M(z)} \tag{75}$$

Hence,

$$\hat{h}_{12} = \frac{(1-r_1)\, U_{1,2}(z)}{(1+r_0)(1+r_1)\, M(z)} \tag{76}$$

Since $U_{1,2}(z)$ is not measurable, we still need an expression for it. To obtain $U_{1,2}(z)$ we should refer to the state equations of the primaries model. The z-transforms of the first two equations of Eq. (10) are

$$z^{-n_1} D_{1,1}(z) = (1+r_0)\ M(z) \tag{77a}$$

$$z^{-n_1} U_{1,1}(z) = r_1\ D_{1,1}(z) + (1-r_1)\ U_{1,2}(z) \tag{77b}$$

From these, we get

$$U_{1,2}(z) = \frac{z^{-n_1}\ U_{1,1}(z) - (1+r_0)r_1\ z^{n_1}\ M(z)}{(1-r_1)} \tag{78}$$

Substituting Eq. (78) into (76) for $U_{1,2}(z)$, we have

$$\hat{h}_{12} = \frac{z^{-n_1}\ U_{1,1}(z)}{(1+r_0)(1+r_1)\ M(z)} - \frac{r_1\ z^{n_1}}{(1+r_1)} \tag{79}$$

Rewriting the first two equation in Eq (5) in z-transformation,

$$z^{-n_1}\ D_1(z) = -r_0\ U_1(z) + (1+r_0)\ M(z) \tag{80a}$$

$$z^{-n_1}\ U_1(z) = r_1\ D_1(z) + (1-r_1)\ U_2(z) \tag{80b}$$

From these, we get

$$U_2(z) = \frac{(z^{-n_1} + r_0 r_1\ z^{n_1})\ U_1(z) - (1+r_0)r_1\ z^{n_1}\ M(z)}{(1-r_1)} \tag{81}$$

Thus, the first two terms in the brackets of Eq. (58) are expressible in terms of known and measurable quantities and $U_{1,1}(z)$. We write Eq. (58) as

$$U_{1,1}(z) = U_1(z) - \left[-r_0\ \hat{h}_{11}\ U_1(z) - r_1\ \hat{h}_{12}\ U_2(z) + \gamma(z)\right], \tag{82}$$

where

$$\gamma(z) = -r_2\ \hat{h}_{13}\ U_3(z) - \ldots - r_{K-1}\ \hat{h}_{1K}\ U_K(z) \tag{83}$$

Substituting Eqs. (60), (79) and (81) into Eq. (82), we have

$$U_{1,1}(z) = \frac{(1-2r_1^2 - r_0 r_1^3\, z^{2n_1})}{(1-r_1^2)}\, U_1(z) + \frac{(1+r_0) r_1^3\, z^{2n_1}}{(1-r_1^2)}\, M(z)$$

$$-\left[\frac{r_1^2}{(1-r_1^2)} - \frac{(r_0+r_1\, z^{-2n_1})}{(1+r_0)(1-r_1^2)}\,\frac{U_1(z)}{M(z)}\right] U_{1,1}(z) - \gamma(z) \tag{84}$$

Finally, solving Eq. (84) for $U_{1,1}(z)$, we have the following equation :

$$U_{1,1}(z) = \frac{(1-2r_1^2 - r_0 r_1^3\, z^{2n_1})\, U_1(z) + (1+r_0) r_1^3\, z^{2n_1}\, M(z)}{1 - \dfrac{(r_0+r_1\, z^{-2n_1})}{(1+r_0)}\,\dfrac{U_1(z)}{M(z)}} - \delta(z) \tag{85}$$

where $\delta(z)$ is a residual term which is deduced from Eq. (84). Substituting Eqs. (64) and (65) into Eq. (85), we obtain

$$Y_1(z) = \frac{(1-2r_1^2 - r_0 r_1^3\, z^{2n_1})\, Y(z) + (1-r_0^2) r_1^3\, z^{2n_1}\, M(z)}{1 - \dfrac{(r_0+r_1\, z^{-2n_1})}{(1-r_0^2)}\,\dfrac{Y(z)}{M(z)}} - \eta(z) \tag{86}$$

where

$$\eta(z) = (1-r_0)\, \delta(z) \tag{87}$$

Now, let

$$F_2(z) = \frac{(1-2r_1^2 - r_0 r_1^3\, z^{2n_1})\, Y(z) + (1-r_0^2) r_1^3\, z^{2n_1}\, M(z)}{1 - \dfrac{(r_0+r_1\, z^{-2n_1})}{(1-r_0^2)}\,\dfrac{Y(z)}{M(z)}} \tag{88}$$

Then

$$F_2(z) = Y_1(z) + \eta(z) \tag{89}$$

Let the relation in Eq. (88) be denoted by F_2 . Then, F_2 is the inverse filter we were looking for. $F_2(z)$ is a better approximation of $Y_1(z)$ than $F_1(z)$ given in Eq. (68). For the simulation purposes, our expression for $F_2(z)$ in Eq. (85) is not so useful because of its complicated structure. In the following section we will show that filter F_2 is just equivalent to the successive application of filter F_1 in two stages. From this, a recursive method to generate $F_n(z)$, $n = 2,1,\ldots,K$, which are the successive approximations of $Y_1(z)$ with $F_K(z) = Y_1(z)$, will be developed in Section IX.

VIII. Effects of Filters F_1 and F_2

Let R(z) be the transfer function of the system of layered media. It is given by

$$R(z) = \frac{Y(z)}{M(z)} \tag{90}$$

Then, Eq. (68) for filter F_1 can be written as

$$F_1(z) = \frac{R(z)}{1 - \frac{r_0}{(1-r_0^2)} R(z)} M(z) \tag{91}$$

If the input is an impulse, then M(z) = 1 and Eq. (91) reduces to

$$F_2(z) = \frac{R(z)}{1 - \frac{r_0}{(1-r_0^2)} R(z)} \tag{92}$$

Now, we define another transfer function G(z) as

$$G(z) = R(z) + r_0 \tag{93}$$

The additional term r_0 represents the direct reflection of the impulse at the surface at time zero (Figure 11a). Notice that R(z) does not include this direct reflection because in our state space model Y(z) was defined as $Y(z) = (1-r_0)\, U_1(z)$ and it excluded the direct reflection term $r_0\, M(z)$ (see Eq. (3)). The expression for filter F_1 in terms of G (z) is obtained just by substituting Eq. (93) for R(z) into Eq. (91) or (92). In a similar manner, we define $R_1(z)$ and $G_1(z)$ to be transfer functions of the subsystem of the layered media below the 1st layer such that

$$G_1(z) = R_1(z) + r_1 \tag{94}$$

This is depicted in Figure 11b. Then, observing the ray diagram in Figure 11c, we can write

$$P(z) = \frac{R(z)}{(1-r_0)} z^{-n_1} \tag{95a}$$

$$Q(z) = (1+r_0) z^{n_1} - \frac{r_0 z^{n_1}}{(1-r_0)} R(z) \tag{95b}$$

$$G_1(z) = \frac{P(z)}{Q(z)} \tag{95c}$$

From these we get the relation between $G_1(z)$ and $R(z)$,

$$G_1(z) = \frac{z^{-2n_1} R(z)}{(1-r_0^2) - r_0 R(z)} \tag{96a}$$

or

$$R(z) = (1-r_0^2) \frac{z^{2n_1} G_1(z)}{1 + r_0 z^{2n_1} G_1(z)} \tag{96b}$$

Substituting Eq. (96b) into Eq. (91), we have

$$F_1(z) = (1-r_0^2) z^{2n_1} G_1(z) M(z) \tag{97a}$$

If the input is an impulse, then $M(z) = 1$, and

$$F_1(z) = (1-r_0^2) z^{2n_1} G_1(z) \tag{97b}$$

This result indicates the fact that <u>the output of the filter $F_1(z)$ is the same as the output of the system of the layered media</u> just below the 1st layer, which is observed at the surface. This is a rather interesting result because the impulse response (or transfer function) of this subsystem can be obtained from the output of the filter, i.e., from eq. (97b),

$$G_1(z) = \frac{z^{-2n_1}}{(1-r_0^2)} F_1(z) \tag{98}$$

This result also indicates that <u>filter F_1 eliminates all the multiple reflections which are ever reflected off of the surface</u>, When the reflection coefficient of the surface is relatively large, the multiple reflections removed by filter F_1 are those which have significant magnitudes and the effect of the filter is especially remarkable.

Filter F_2 given in Eq. (88) is expressed in terms of R(z) as

$$F_2(z) = \frac{(1-2r_1^2 - r_0 r_1^3 z^{2n_1}) R(z) + (1-r_0^2) r_1^3 z^{2n_1}}{1 - \frac{(r_0+r_1 z^{-2n_1})}{(1-r_0^2)} R(z)} M(z) \tag{99}$$

Substituting Eq. (96b) for R(z) into Eq. (99), we get

$$F_2(z) = (1-r_0^2) z^{2n_1} \frac{(1-2r_1^2) G_1(z) + r_1^3}{1 - r_1 G_1(z)} M(z) \tag{100}$$

Again, substituting Eq. (94) into Eq. (100) for $G_1(z)$, we obtain the following equation,

$$F_2(z) = (1-r_0^2) z^{2n_1} \left[r_1 + \frac{R_1(z)}{1 - \frac{r_1}{(1-r_1^2)} R_1(z)} \right] M(z) \tag{101}$$

Here, $R_1(z)$ is, as we defined in Eq. (94), the transfer function of the subsystem of the layered media below the 1st layer, excluding the direct reflection term r_1 at time zero. The second term in the brackets in Eq. (101) is exactly of the same form as filter F_1, hence this term represents the filtered output of the subsystem below the 1st layer. The first term $(1-r_0^2) z^{2n_1} r_1 M(z)$ in Eq. (101) represents the primary reflection associated with the 1st interface of the media. Hence, $F_2(z)$ can be obtained by applying filter F_1 twice successively, first to the given system and then to the subsystem below the 1st layer; i.e.,

(1) apply F_1 to Y(z) to get $F_1(z)$,

(2) then, compute $R_1(z)$ by Eqs. (98) and (94) from $F_1(z)$ and apply F_1 again to $R_1(z)$. $F_2(z)$ is then obtained by substituting this result into Eq. (101).

Now, if we define $G_2(z)$ and $R_2(z)$ to be the transfer functions of the subsystem below the second layer in the same way as we defined $G_1(z)$ and $R_1(z)$, such that

$$G_2(z) = R_2(z) + r_2 , \tag{102}$$

then, it is not difficult to show that

$$\frac{R_1(z)}{1 - \frac{r_1}{(1-r_1^2)} R_1(z)} = (1-r_1^2)\, z^{2n_2}\, G_2(z) \tag{103}$$

Equation (103) is analogous to Eq. (97b). Substituting this into Eq. (101), we have

$$F_2(z) = \left[(1-r_0^2) r_1\, z^{2n_1} + (1-r_0^2)(1-r_1^2)\, z^{2n_1+2n_2}\, G_2(z)\right] M(z) \tag{104}$$

This result shows that $F_2(z)$ is just the output of the subsystem below the 2nd layer, which is observed on the surface, plus the primary reflections reflected from the 1st interface. This is described in Figure 12. This result also indicates that filter F_2 eliminates all the multiple reflections which are ever reflected down from the surface and the 1st interface.

IX. Recursive Scheme for the Successive Approximation of the Primaries $Y_1(z)$.

In Section VIII we have already recognized that $F_2(z)$ can be obtained in a recursive way applying filter F_1 in two stages, first to the whole system $R(z)$ and then to the subsystem $R_1(z)$ which is obtained directly from the result of the previous step. This procedure can be extended to apply filter F_1 to the consecutive subsystems $R_2(z), R_3(z), \ldots, R_{K-1}(z)$ generating $F_3(z), F_4(z), \ldots, F_K(z)$. The subsystems are computed recursively during this procedure. The outputs $F_1(z), F_2(z), \ldots, F_K(z)$ are then successive approximations to the primaries $Y_1(z)$. It will be shown that in the deterministic situations (i.e., perfect measurements) the final output, $F_K(z)$, represents the pure primaries of the system.

Observe, that to compute $F_1(z)$ we need to know the value of r_0, and to obtain $F_2(z)$, additional knowledge of τ_1 and r_1 is necessary. Likewise, in order to apply the recursive procedure to obtain the third output $F_3(z)$, knowledge of τ_2 and r_2 is required, and in general, to get $F_n(z)$ at the n-th stage of the procedure we need knowledge of τ_{n-1} and r_{n-1}. In this paper we just assume that those values are known in each step of the procedure (by estimation or whatever). In a later paper we shall discuss the estimation of these quantities.

Just as we defined $G_1(z)$, $R_1(z)$ and $G_2(z)$, $R_2(z)$ in Section VIII, let $G_i(z)$, $R_i(z)$ be the transfer functions of the subsystem of the media below the i-th layer; $G_i(z)$ includes the direct reflection term r_i at time zero and $R_i(z)$ is just the same as $G_i(z)$ except that it excludes the direct reflection term; i.e.,

$$G_i(z) = R_i(z) + r_i \qquad (105)$$

$i = 1,2,\ldots,K$. Using the same argument as presented before to obtain Eq. (96b), it is not difficult to show that $R_i(z)$ and $G_{i+1}(z)$ are related as follows :

$$R_i(z) = (1-r_i^2)\,\frac{z^{2n_{i+1}}\,G_{i+1}(z)}{1 + r_i\,z^{2n_{i+1}}\,G_{i+1}(z)} \qquad (106)$$

Let us define $X_i(z)$ to be

$$X_i(z) = \frac{R_i(z)}{1 - \frac{r_i}{(1-r_i^2)} R_i(z)} \tag{107}$$

Then, $X_i(z)$ is the output of the filter F_1 which is applied to the subsystem $R_i(z)$. Substituting Eq. (106) into (107) for $R_i(z)$, we have

$$X_i(z) = (1-r_i^2)\, z^{2n_{i+1}}\, G_{i+1}(z) \tag{108}$$

From Eqs. (105) and (108), we obtain

$$R_{i+1}(z) = \frac{z^{-2n_{i+1}}}{(1-r_i^2)} X_i(z) - r_{i+1} \tag{109}$$

This equation states the important fact that subsystem $R_{i+1}(z)$ can be obtained from the filtered output of subsystem $R_i(z)$. This Eqs. (107) and (109) represent a recursive relation for $R_i(z)$'s, $i = 1,2,\ldots,K$, with $R(z)$ as the *starting value*. *Equations* (97a) and (104) can be written in terms of $R_1(z)$ and $R_2(z)$ as

$$F_1(z) = \left[(1-r_0^2)r_1\, z^{2n_1} + (1-r_0^2)\, z^{2n_1}\, R_1(z)\right] M(z) \tag{110}$$

and

$$F_2(z) = \Big[(1-r_0^2)r_1\, z^{2n_1} + (1-r_0^2)(1-r_1^2)r_2\, z^{2n_1+2n_2}$$

$$+ (1-r_0^2)(1-r_1^2)\, z^{2n_1+2n_2}\, R_2(z)\Big] M(z) \tag{111}$$

Observing the expressions for $F_2(z)$ in Eq. (101), we can see that $F_2(z)$ is obtained if we replace $R_1(z)$ in Eq. (110) by $X_1(z)$, the filtered output of $R_1(z)$. Likewise, we expect that $F_3(z)$ will be obtained if we replace $R_2(z)$ in Eq. (111) by $X_2(z)$. This procedure can be continued to generate $F_4(z), F_5(z), \ldots, F_K(z)$ with the help of the recursive relations given by Eqs. (107) and (109).

Then, $F_{i+1}(z)$ has the following structure;

$$F_{i+1}(z) = \left[\sum_{\ell=1}^{i} \left(\prod_{k=0}^{\ell-1} (1-r_k^2)\, z^{2n_{k+1}} \right) r_\ell + \left(\prod_{k=0}^{i} (1-r_k^2)\, z^{2n_{k+1}} \right) X_i(z) \right] M(z) \tag{112}$$

The first term in the brackets represents the primaries which are associated with the first n interfaces (interfaces 1~n) of the media and the second term represents the filtered output $X_n(z)$ of the subsystem $R_n(z)$ observed at the surface. This is depicted in Figure 13. Since the effect of the filter is to remove all the reflections which are ever reflected down off the top surface of the media (subsystem), $F_n(z)$, as the result of the n-th stage of the procedure is free from all those multiple reflections which are ever reflected down off the surface and the first (n-1) interfaces. If we continue this procedure, after the K-th stage all the multiple reflections will be removed completely from the seismogram and only primary reflections will remain. To see this, observe from Eq. (112) that

$$F_K(z) = \left[\sum_{\ell=1}^{K-1} \left(\prod_{k=0}^{\ell-1} (1-r_k^2)\, z^{2n_{k+1}} \right) r_\ell + \left(\prod_{k=0}^{K-1} (1-r_k^2)\, z^{2n_{k+1}} \right) X_{K-1}(z) \right] M(z) \tag{113}$$

But from Eq. (108)

$$X_{K-1}(z) = (1-r_{K-1}^2)\, z^{2n_K}\, G_K(z) \tag{114}$$

where the subsystem $G_K(z)$ below the K-th layer is just the basement and

$$G_K(z) = r_k \,. \tag{115}$$

Hence,

$$F_K(z) = \left[\sum_{\ell=1}^{K} \left(\prod_{k=0}^{\ell-1} (1-r_k^2)\, z^{2n_{k+1}} \right) r_\ell \right] M(z) \tag{116}$$

which represents the pure primaries of the system.

Summarizing the recursive procedure :

Starting Equation

$$X_0(z) = \frac{R(z)}{1 - \dfrac{r_0}{(1-r_0^2)} R(z)} \tag{117}$$

Recursive Relations

$$R_i(z) = \frac{z^{-2n_i}}{(1-r_{i+1}^2)} X_{i-1}(z) - r_i \tag{118}$$

$$X_i(z) = \frac{R_i(z)}{1 - \dfrac{r_i}{(1-r_i^2)} R_i(z)} \tag{119}$$

Output Equations

$$F_1(z) = X_0(z)\, M(z) \tag{120}$$

$$F_{i+1}(z) = \left[\sum_{\ell=1}^{i} \left(\prod_{k=0}^{\ell-1} (1-r_k^2)\, z^{2n_{k+1}} \right) r_\ell + \left(\prod_{k=0}^{i} (1-r_k^2)\, z^{2n_{k+1}} \right) X_i(z) \right] M(z) \tag{121}$$

where $i = 1,2,\ldots,K-1$.

Figure 14 depicts filtered outputs $F_2(z)$ and $F_3(z)$ for the three layer example which are obtained by the recursive procedure above (This is continuation of Figure 10). The last result, $F_3(z)$, which is shown in Figure 14(b) exhibits the three pure primaries of the system. Since, in this example, the considerable amount of multiple reflections was removed already in the first stage (see Figure 10(b)), $F_2(z)$ and $F_3(z)$ in Figure 14 shows little changes in improvement.

This is due to the fact that the surface reflection coefficient, r_0, in this example is much greater than r_1,r_2 and r_3 in magnitudes. Figure 15 depicts another example of the recursive filtering. Figure 15(a) is the complete response of a six layer media. The values of the reflection coefficients and the one-way travel times of the media used in this example are ;

$$r_0 = 0.68 \ , \ r_1 = 0.40 \ , \ r_2 = -0.32 \ , \ r_3 = 0.53,$$

$$r_4 = -0.78 \ , \ r_5 = 0.71 \ , \ r_6 = 0.65 \ , \text{ and}$$

$$\tau_1 = 0.07 \ , \ \tau_2 = 0.04 \ , \ \tau_3 = 0.115, \ \tau_4 = 0.09,$$

$$\tau_5 = 0.035, \ \tau_6 = 0.13 \ ,$$

The same input given in Figure 9 is used in this example. Figures 15 (b)-(g) show successive filtered outputs, $F_1(z)$, $F_2(z)$, ..., $F_6(z)$, which are generated by our recursive filtering procedure. Again, $F_6(z)$ in Figure 15(g) represents the primaries of the six layer media.

X. Conclusions

We have presented a simple inverse filter which suppresses a fair amount of the multiple reflections in a synthetic seismogram. This filter requires knowledge of the surface reflection coefficient and the input waveform. The Bremmer series decomposition played a key role in its development. The filter was shown to be especially useful when the surface reflection coefficient is relatively large (as in most geophysical situations) in which case significant multiple reflections are almost completely removed so that the output is a good approximation of the primaries of the layered system.

The recursive filtering method presented herein demonstrates the possibilities that the subsystems of the layered media can be revealed from the seismogram by applying the filter successively and that pure primaries of the system can be obtained thereof. The actual application of this recursive procedure requires the estimation of the reflection coefficients and the oneway travel time in each stage to perform the next recursion; but, this in turn, provides motivation to use this procedure in estimating those quantities. The estimation scheme is now under research. Also, work is in progress on developing a version of this filter that is applicable to noisy seismogram or realistic field seismic data and to the situation when the source waveform is not available.

ACKNOWLEDGEMENT

The work reported on in this paper was performed at the University of Southern California, Los Angeles, California, under National Science Foundation Grant NSF ENG 74-02297 A01, Air Force Office of Scientific Research Grant AFOSR 75-2797, Chevron Oil Field Research Co. Contract-76 and U.S. Geological Survey, Department of the Interior, under USGS Grant No. 14-08-0001-G-553.

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LIST OF FIGURES

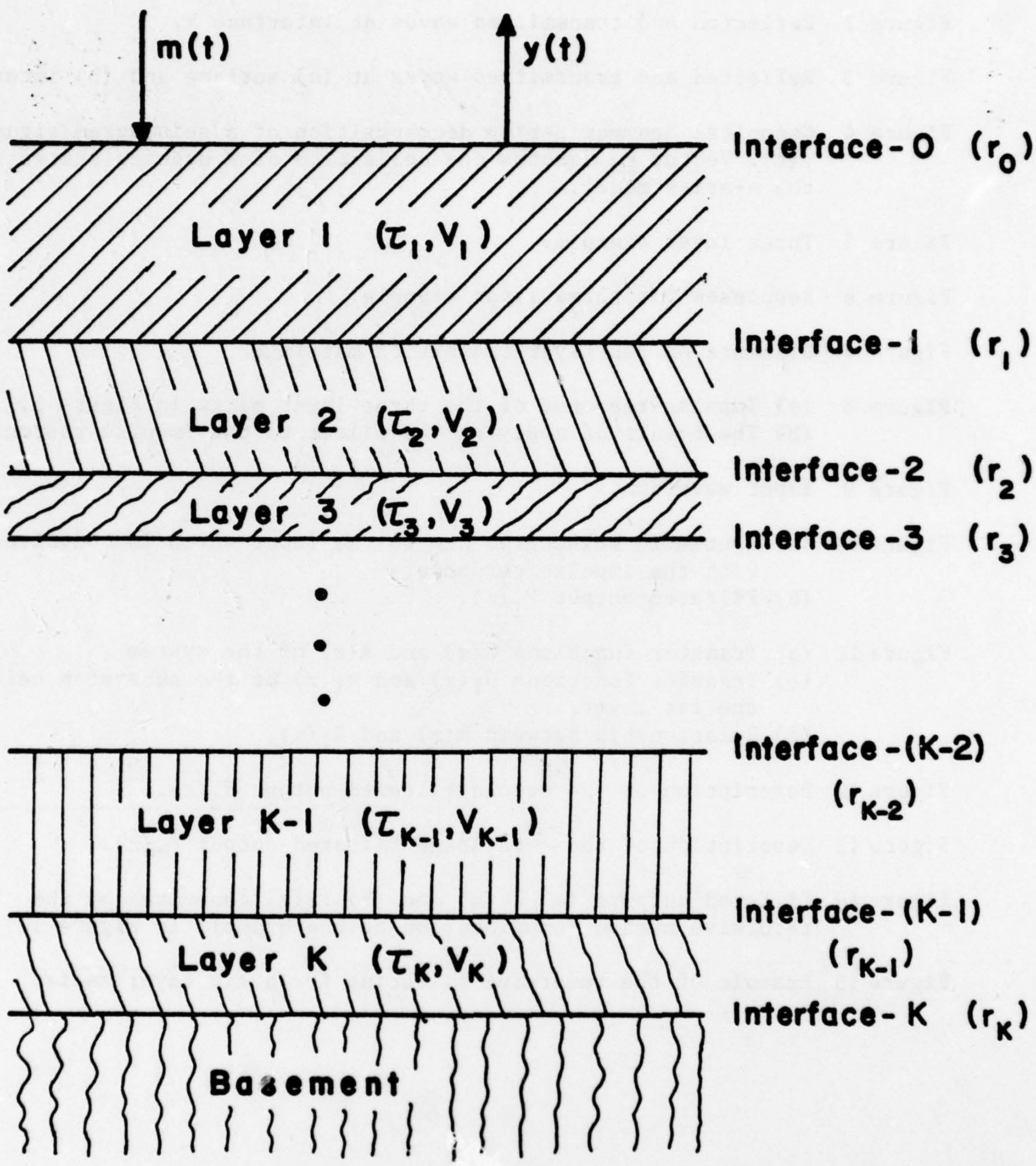

m(t)
y(t)
Interface-0 (r_0)
Layer 1 (τ_1, V_1)
Interface-1 (r_1)
Layer 2 (τ_2, V_2)
Interface-2 (r_2)
Layer 3 (τ_3, V_3)
Interface-3 (r_3)
Interface-(K-2) (r_{K-2})
Layer K-1 (τ_{K-1}, V_{K-1})
Interface-(K-1) (r_{K-1})
Layer K (τ_K, V_K)
Interface-K (r_K)
Basement

Figure 1

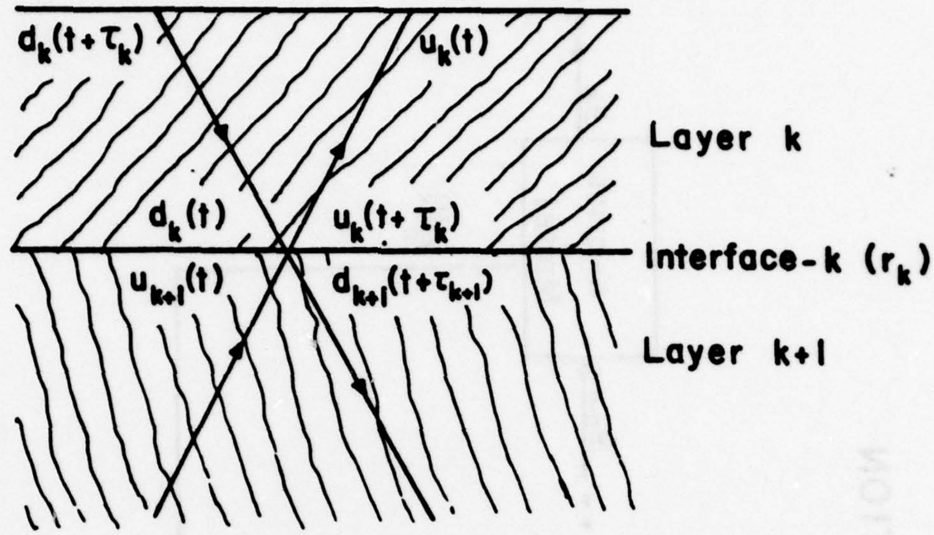


Figure 2

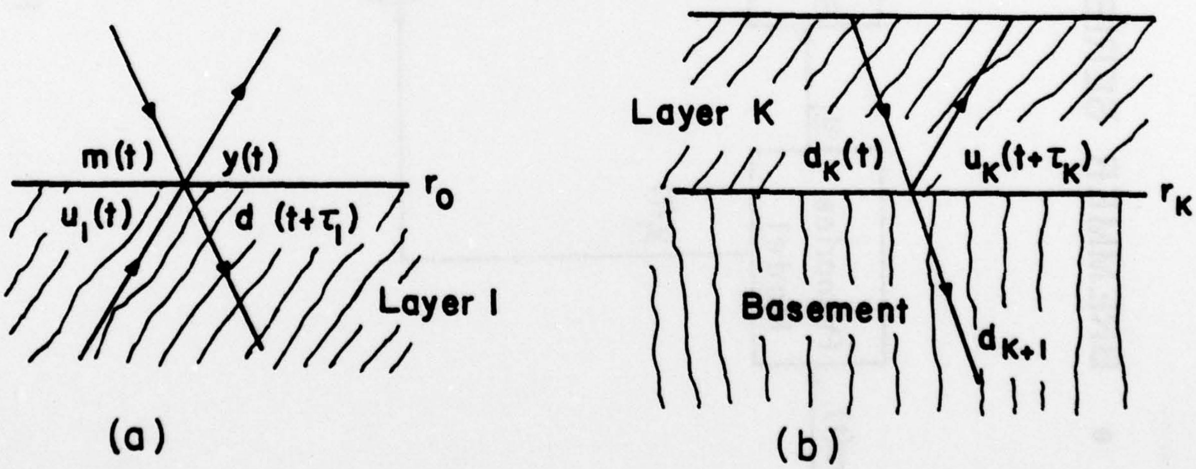


Figure 3

- **BREMMER SERIES DECOMPOSITION**

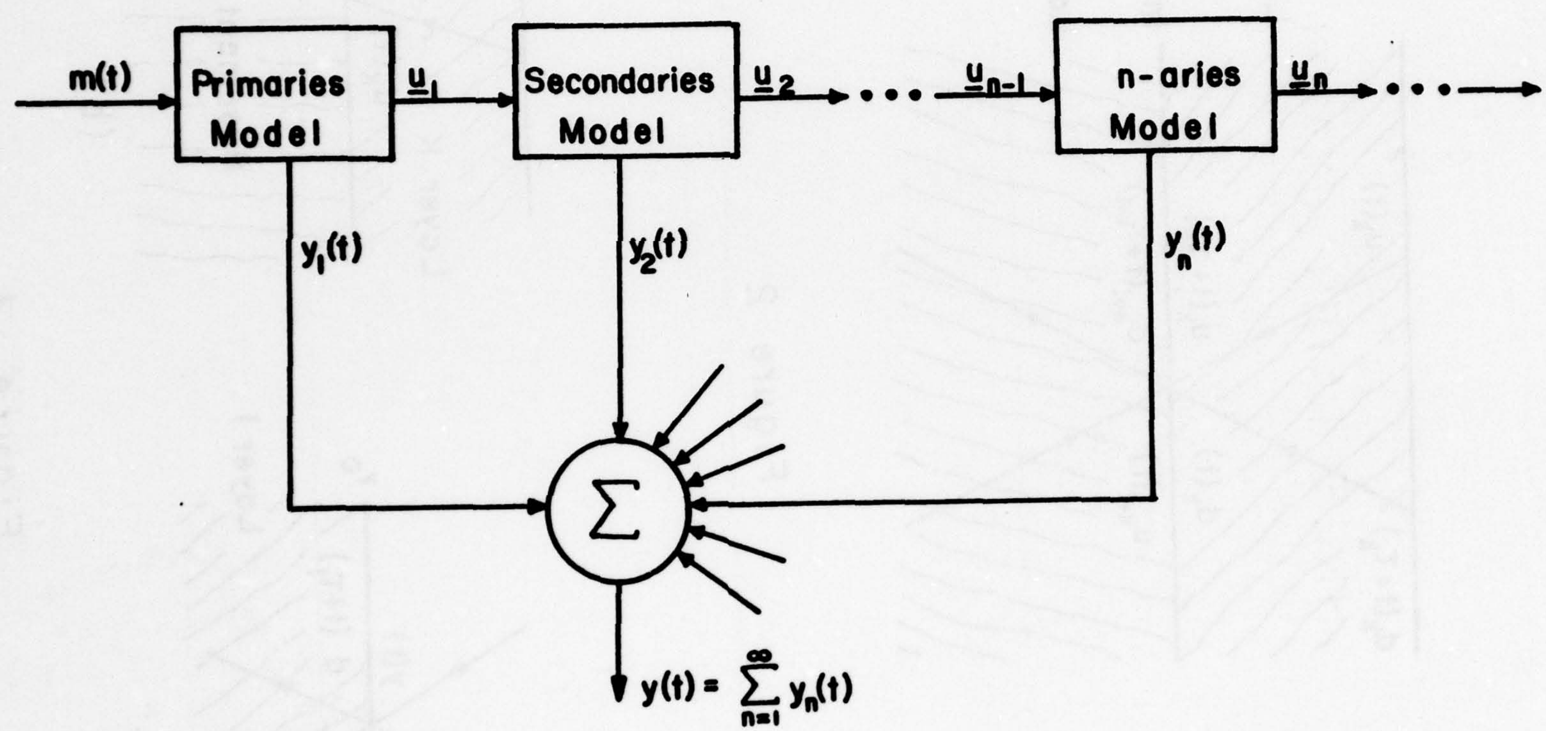


Figure 4

THREE LAYER EXAMPLE

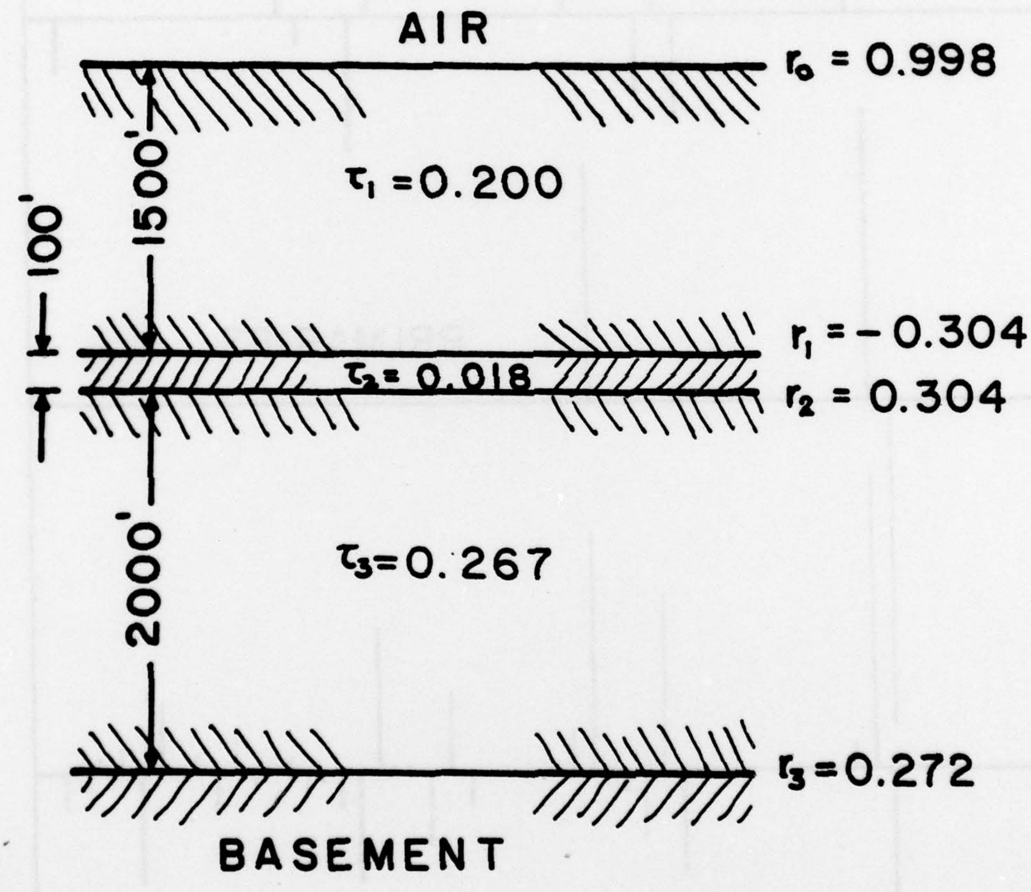
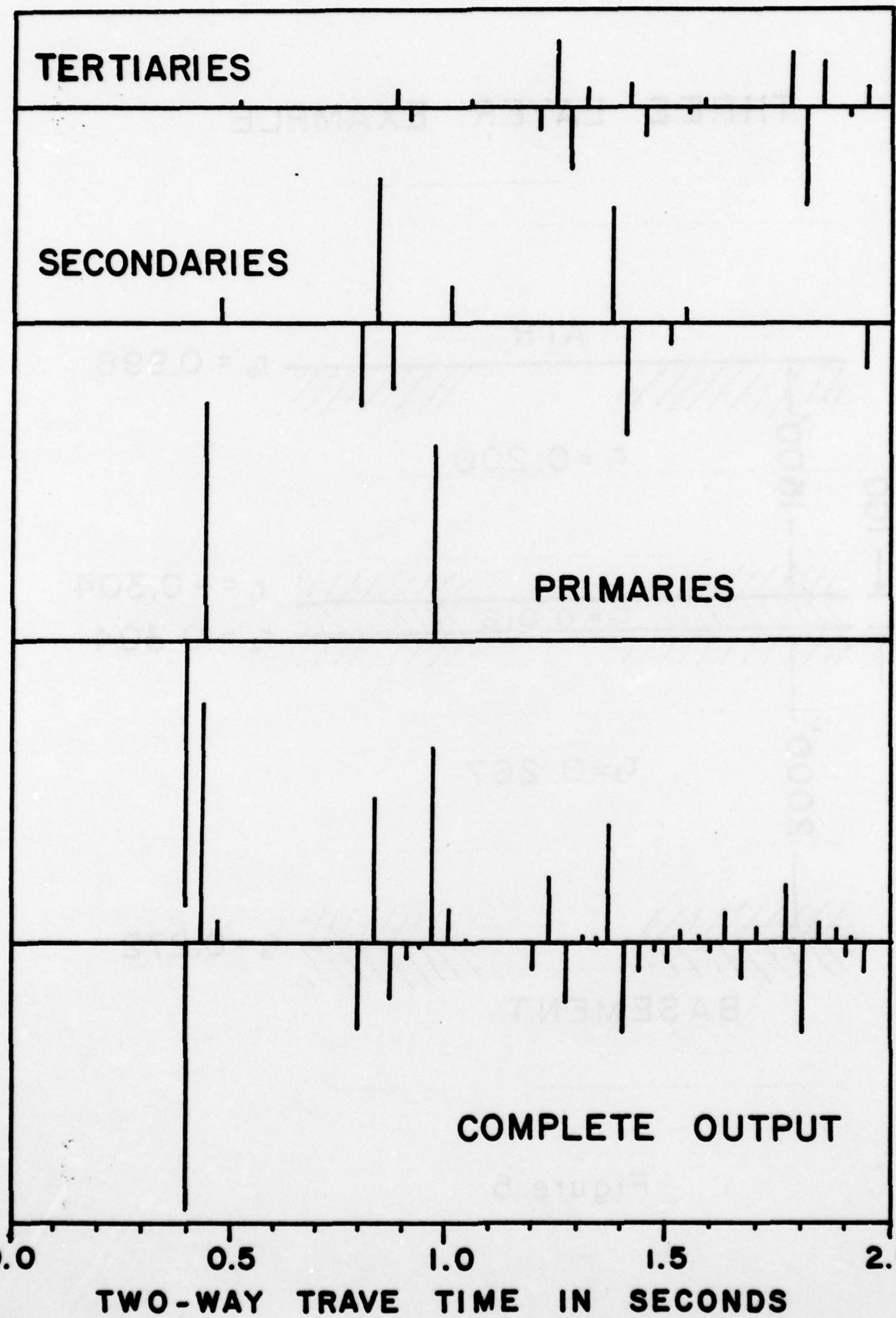
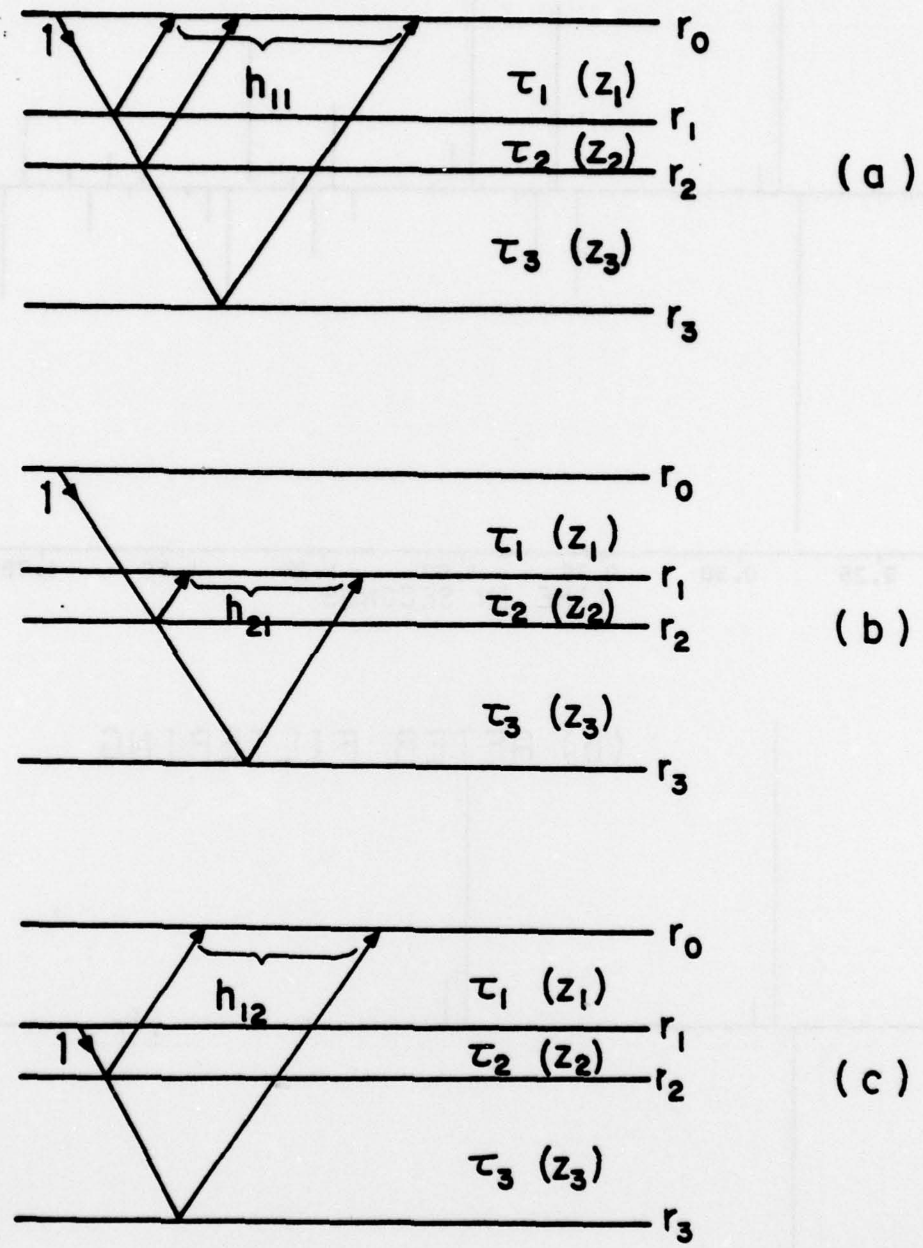


Figure 5

Figure 6


TERTIARIES
SECONDARIES
PRIMARIES
COMPLETE OUTPUT
0.0
0.5
1.0
1.5
2.0
TWO-WAY TRAVE TIME IN SECONDS



1
h_{11}
τ_1 (z_1)
τ_2 (z_2)
τ_3 (z_3)
r_0
r_1
r_2
r_3
(a)
1
h_{21}
τ_1 (z_1)
τ_2 (z_2)
τ_3 (z_3)
r_0
r_1
r_2
r_3
(b)
1
h_{12}
τ_1 (z_1)
τ_2 (z_2)
τ_3 (z_3)
r_0
r_1
r_2
r_3
(c)

Figure 7

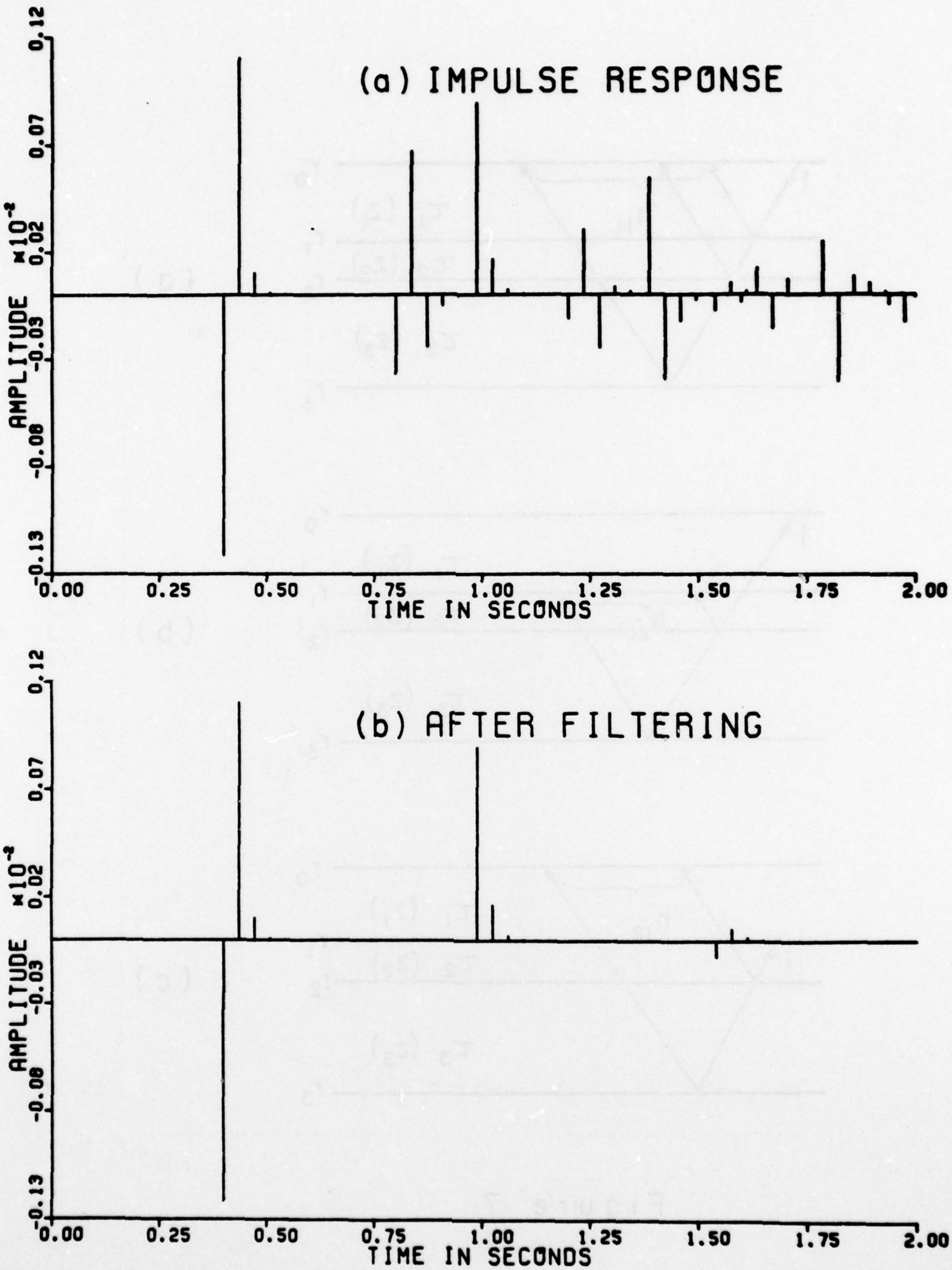

(a) IMPULSE RESPONSE
AMPLITUDE ×10⁻²
0.12
0.07
0.02
-0.03
-0.08
-0.13
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
TIME IN SECONDS
(b) AFTER FILTERING
AMPLITUDE ×10⁻²
0.12
0.07
0.02
-0.03
-0.08
-0.13
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
TIME IN SECONDS

Figure 8

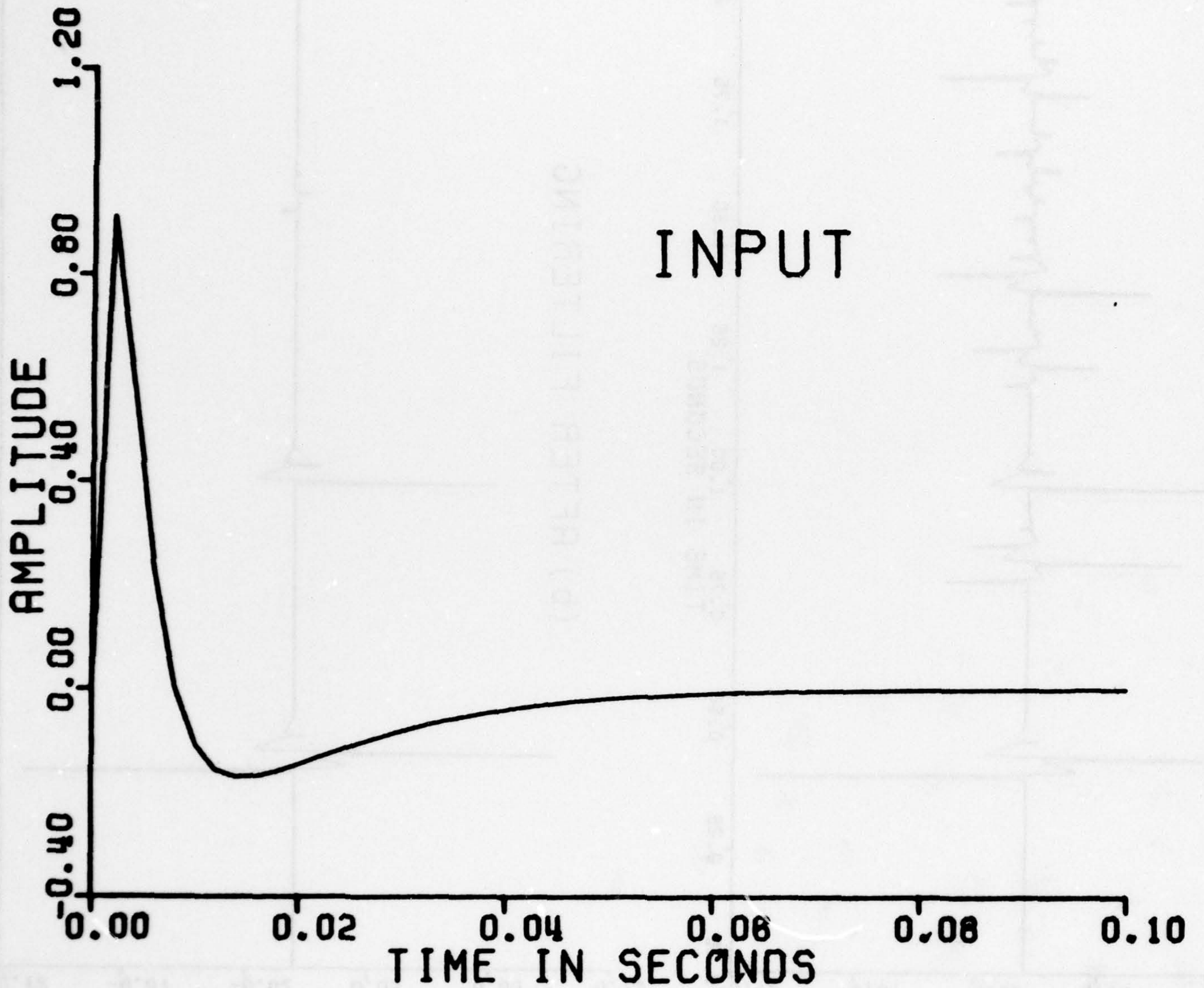

INPUT
AMPLITUDE
1.20
0.80
0.40
0.00
-0.40
0.00
0.02
0.04
0.06
0.08
0.10
TIME IN SECONDS

Figure 9

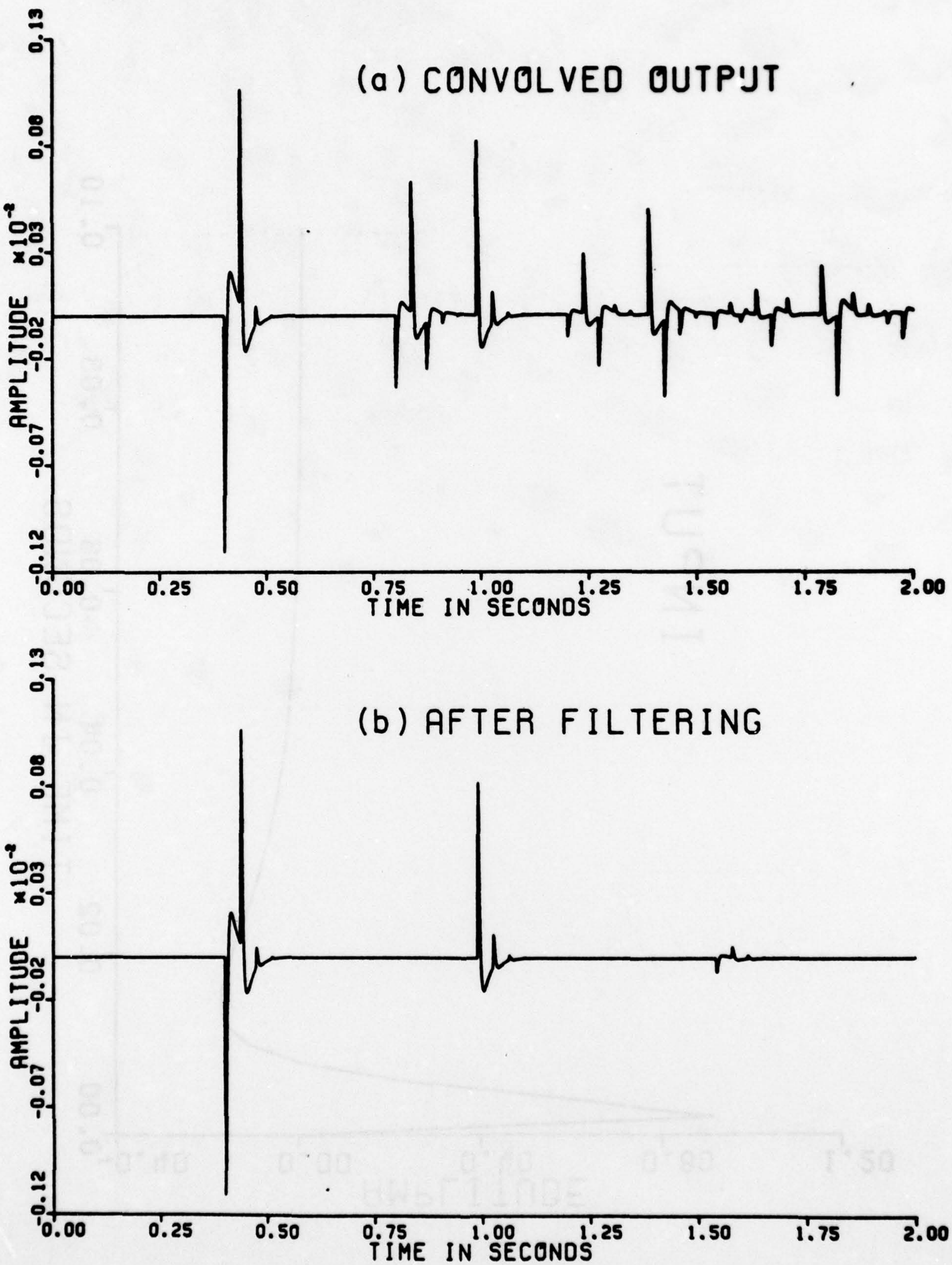
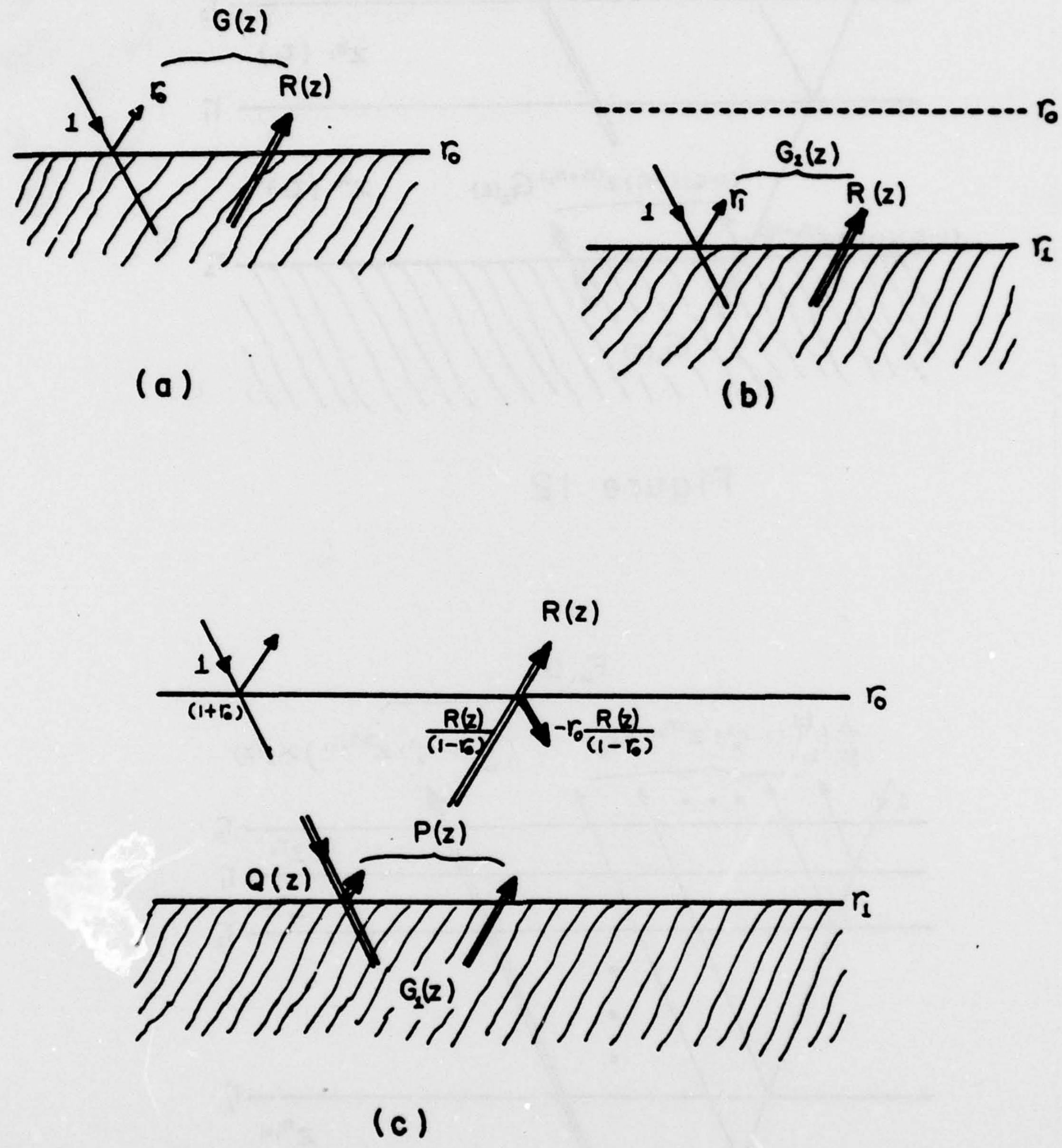

(a) CONVOLVED OUTPUT
AMPLITUDE ×10^-2
0.13
0.08
0.03
-0.02
-0.07
-0.12
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
TIME IN SECONDS
(b) AFTER FILTERING
AMPLITUDE ×10^-2
0.13
0.08
0.03
-0.02
-0.07
-0.12
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
TIME IN SECONDS

Figure 10



G(z)
1
r_0
R(z)
r_0
(a)
r_0
$G_1(z)$
1
r_1
R (z)
r_1
(b)
R(z)
1
r_0
$(1+r_0)$
$\frac{R(z)}{(1-r_0)}$
$-r_0\frac{R(z)}{(1-r_0)}$
P(z)
Q(z)
r_1
$G_1(z)$
(c)

Figure 11

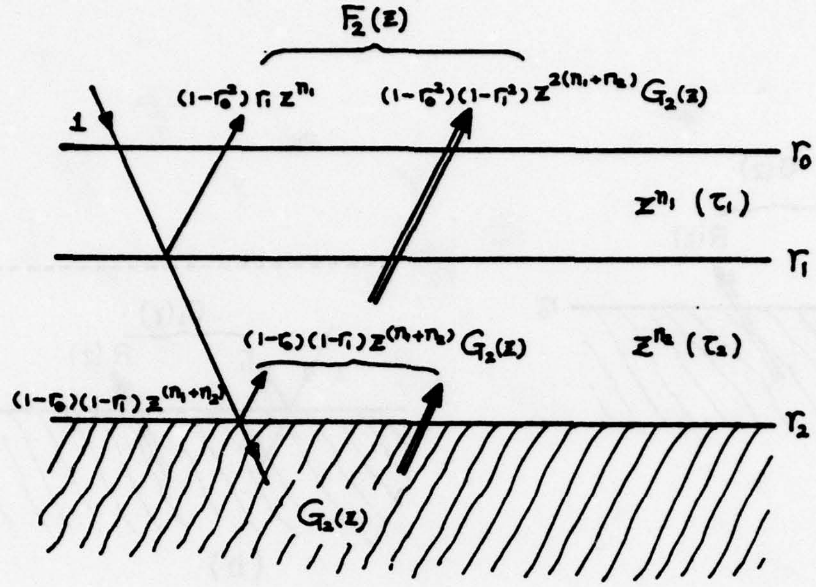


Figure 12

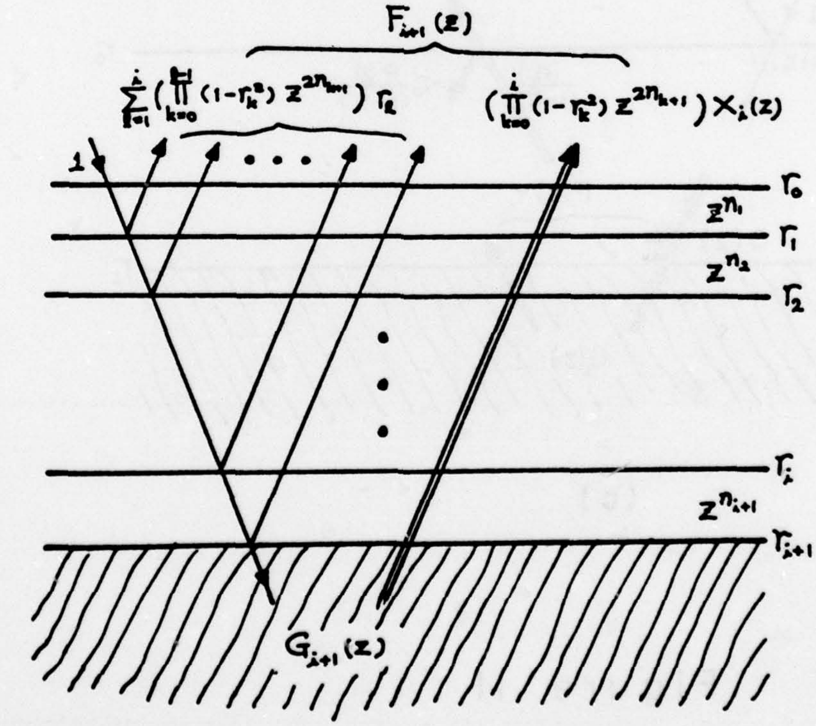


Figure 13

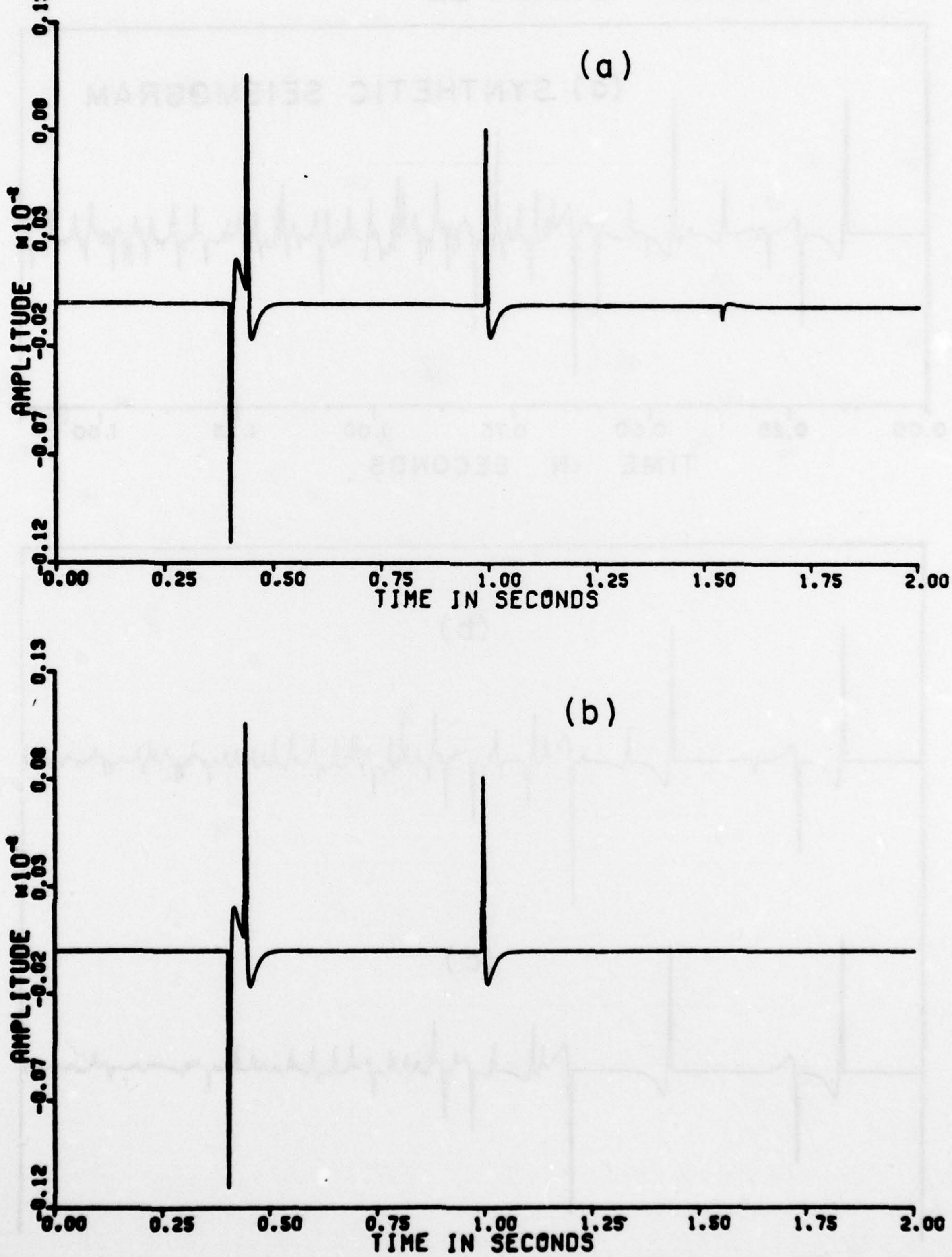

(a)
AMPLITUDE $\times 10^{-4}$
0.13
0.08
0.03
-0.02
-0.07
-0.12
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
TIME IN SECONDS
(b)
AMPLITUDE $\times 10^{-4}$
0.13
0.08
0.03
-0.02
-0.07
-0.12
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
TIME IN SECONDS

Figure 14

SIX LAYER EXAMPLE

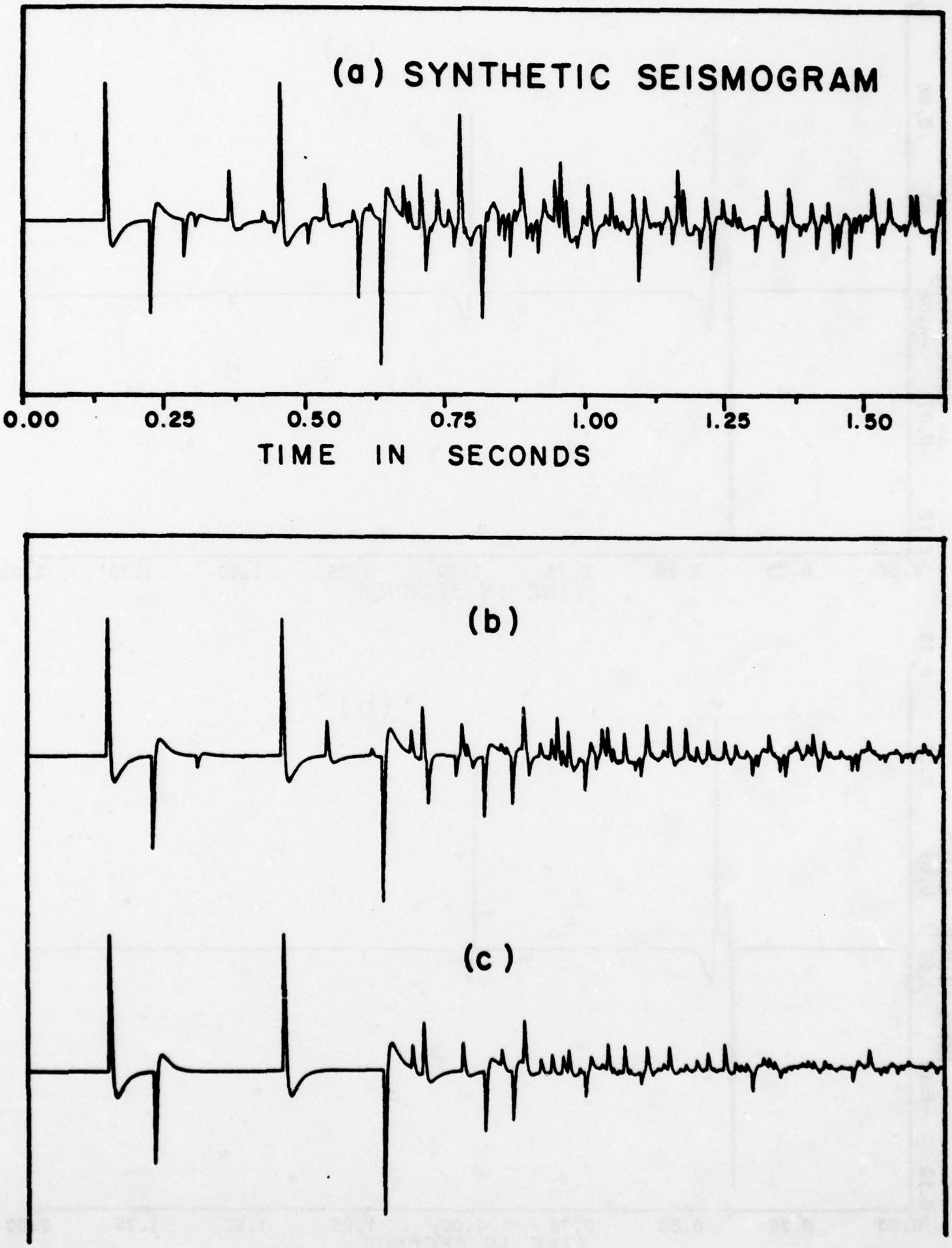


Figure 15

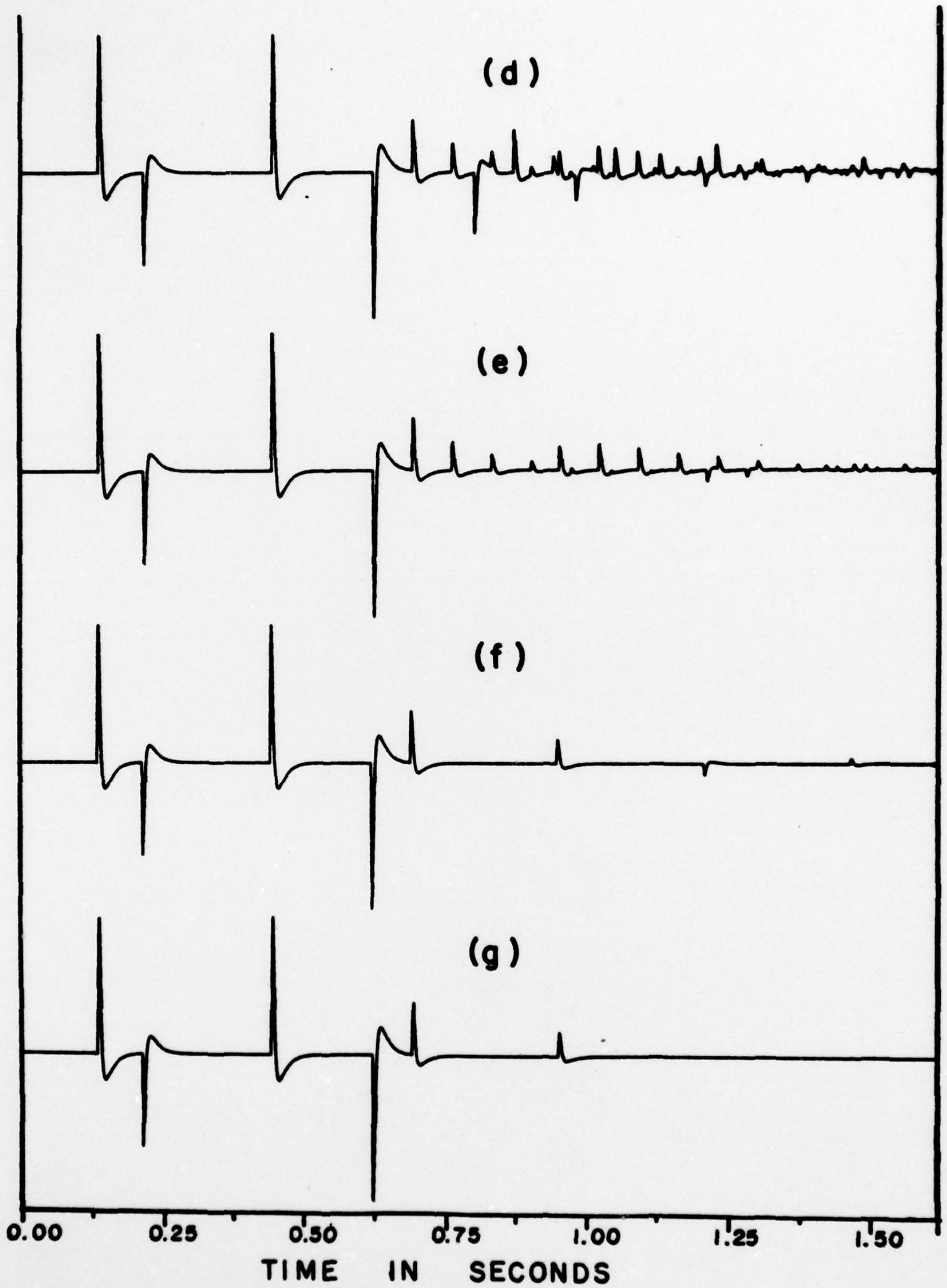


Figure 15 (continued)